Doubly semi-equivelar maps on torus and Klein bottle

Anand K Tiwari[1], Amit Tripathi[2], Yogendra Singh[1], and Punam Gupta[3]

[1]Department of Applied Science, Indian Institute of Information Technology, Allahabad 211 015, India,
E-mail: {anand, rss2018503}@iiita.ac.in
[2]Department of Applied Science & Humanities, Rajkiya Engineering College, Banda -210201, India,,
E-mail: amittripathi@recbanda.ac.in
[3]Department of Mathematics & Statistics, Dr. Hari Singh Gour Vishwavidyalaya, Sagar 470 003, India,,
E-mail: punam2101@gmail.com

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Abstract

A tiling of the Euclidean plane, by regular polygons, is called 2-uniform tiling if it has two orbits of vertices under the action of its symmetry group. There are 20 distinct 2-uniform tilings of the plane. Plane being the universal cover of torus and Klein bottle, it is natural to ask about the exploration of maps on these two surfaces corresponding to the 2-uniform tilings. We call such maps as doubly semi-equivelar maps. In the present study, we compute and classify (up to isomorphism) doubly semi-equivelar maps on torus and Klein bottle. This classification of semi-equivelar maps is useful in classifying a category of symmetrical maps which have two orbits of vertices, named as 2-uniform maps.

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1 Introduction

Equivelar and semi-equivelar maps are generalizations of the maps on the surfaces of well known Platonic solids and Archimedean solids to the closed surfaces other than the 2-sphere, respectively. A substantial literature is available for such maps, see [1, 2, 3, 4, 5, 6, 7, 8].

Tilings of the plane are a great source of polyhedral maps on the surfaces of torus and Klein bottle, as the plane is the universal cover of these two surfaces. A tiling of the plane, by regular polygons, is called a k-uniform tiling if it has k orbits of vertices under its symmetry. The k-uniform tilings have been completely enumerated for $k \leq 6$. There are 11 1-uniform, 20 2-uniform, 61 3-uniform, 151 4-uniform, 332 5-uniform and 673 6-uniform tilings on the plane. For a detailed study on such tilings, readers are referred to see [9, 10, 11].

The 11 1-uniform tilings of the plane are also called Archimedean tilings. Out of these, 3 are regular and 8 are semi-regular tilings. The 3 regular tilings provide equivelar maps of types $[3^6]$, $[4^4]$, $[6^3]$ and 8 semi-regular tilings provide semi-equivelar maps of types $[3^4, 6]$, $[3^3, 4^2]$, $[3^2, 4, 3, 4]$, $[3, 4, 6, 4]$, $[3, 6, 3, 6]$, $[3, 12^2]$, $[4, 6, 12]$ and $[4, 8^2]$, on torus and Klein bottle. Altshuler [12] has given a construction for a map of the type $[3^6]$ and $[6^3]$ on the torus. Kurth [13] has enumerated maps of the types $[3^6]$, $[4^4]$ and $[6^3]$ on the torus. In [2], Datta band Nilakantan classified map of type $[3^6]$ and $[4^4]$ on at most 11 vertices. In continuation of this, Datta and Upadhyay [14] classified these type of maps for n vertices with $12 \leq n \leq 15$. In [15], Brehm and Kuhnel have classified these three types equivelar maps on the torus using a different approach. In [16], Tiwari and Upadhyay have classified the 8 types semi-equivelar maps on at most 20 vertices. Recently, Maity and Upadhyay [17] have presented a way to classify the eight types of semi-equivelar maps on the torus for arbitrary number of vertices.

Analogues to the Archimedean tilings, here we initiate the theory of maps on torus and Klein bottle corresponding to the 2-uniform tilings. We call such maps as doubly semi-equivelar map(s) or briefly DSEM(s). The present work provides a new class of polyhedra which have two classes of vertices in terms of the arrangement of polygons around the vertices. Polyhedra play an important role in human life. It has extensive application in ornament designing, architectural designing, cartography, computer graphics etc., see [18, 19, 20].

This article is organized as follows: In Sec 2, we give basic definitions and notations used in the present work. In Sec 3, we define doubly semi-equivelar map (DSEM) and describe a methodology to enumerate a doubly semi-equivelar map on torus and Klein bottle. In Sec 4, we compute and classify DSEMs on torus and Klein bottle. In Sec 5, we present the results obtained from the computation and classification. A tabular form of the results is shown in Table 5. In Sec 6, we present discussions and future scope of the DSEMs followed by some concluding remarks.

2 Basic definitions and notations

For graph theory related terminologies, we refer [21]. A p-cycle, denoted as C_p, is a 2-regular graph with p vertices. We denote C_p explicitly as $C_p(v_1, \ldots, v_p)$, where the vertex set $V(C_p) = \{v_1, \ldots, v_p\}$ and edge set $E(C_p) = \{v_1v_2, \ldots, v_{n-1}v_n, v_nv_1\}$.

A surface (closed surface) F is a connected, compact 2-manifold without boundary. A surface F is either sphere, sphere with g handles (also called orientable surface of genus g, denoted as S_g) or sphere with g cross caps (also called non-orientable surface of genus g, denoted as N_g). To a surface, we associate a unique integer called its Euler characteristic χ and is defined as $\chi(S_g) = 2 - 2g$ and $\chi(N_g) = 2 - g$. The surfaces S_1 and N_2 of Euler characteristic 0 are called torus and Klein bottle, respectively.

An embedding of a connected, simple graph G into a surface F is called 2-cell embedding if the closure of each connected component of $F \setminus G$ is a 2-disk D_p. These components are called faces of the embedding. The vertices and edges of G are called the vertices and edges of the embedding. A map (polyhedral) M on a surface F is a 2-cell embedding such that the non-empty intersection of any two faces is either a vertex or an edge [22]. The face size of a map M is p, if p is the largest positive integer such that M has a face D_p.

Two maps M_1 and M_2, with vertex sets $V(M_1)$ and $V(M_2)$ respectively, are said to be isomorphic if there is a bijective map $f : V(M_1) \to V(M_2)$ which preserves the incidence of edges and incidence of faces. An isomorphism from a map M to itself is also called an automorphism. A collection $Aut(M)$ of all the automorphisms of a map M forms a group under the composition of maps, called the automorphism group of M. A map M is called vertex-transitive if it has a unique orbit of vertices under the action of $Aut(M)$.

The face-sequence [7] of a vertex v, denoted as $f\text{-}seq(v)$, in a map M is a finite cyclic sequence $(p_1^{n_1}, \ldots, p_k^{n_k})$, where $p_1, \ldots, p_k \geq 3$ and $n_1, \ldots, n_k \geq 1$, such that the face cycle at v is $(D_{p_1}, \ldots (n_1\text{times}), \ldots, D_{p_k}, \ldots (n_k\text{times}))$. A map is called semi-equivelar of type $[p_1^{n_1}, \ldots, p_k^{n_k}]$ if the face-sequence of each vertex is $(p_1^{n_1}, \ldots, p_k^{n_k})$. A semi-equivelar map of type $[p^n]$ is also called equivelar map.

Let $(D_{p_1}, \ldots, D_{p_k})$ be the face cycle at a vertex v in a map M. Let C_{p_i} denote the boundary cycles of these D_{p_i}. Then the link of v, denoted as $\mathrm{lk}(v)$, is a cycle in M consisting of all the vertices of these C_{p_i}'s except v and all the edges of these C_{p_i}'s except which has one end vertex v. If v is a vertex with $\mathrm{lk}(v) = C_k(v_1, \ldots, v_k)$, the face-sequence of $\mathrm{lk}(v)$ is a cyclically ordered sequence $(f\text{-}seq(v_1), \ldots, f\text{-}seq(v_k))$.

Let v be a vertex with the face sequence $(p_1^{n_1}, \ldots, p_k^{n_k})$. The combinatorial curvature of v, denoted by $\phi(v)$, is defined as $\phi(v) = 1 - (\sum_{i=1}^{k} n_i)/2 + (\sum_{i=1}^{k} n_i/p_i))$.

3 Definition of the problem and description of method

Let M be a map with two distinct face-sequences f_1 and f_2. We say that M is a doubly semi-equivelar map, in short DSEM, if (i) the sign of $\phi(v)$ is same for all $v \in M$ (ii) vertices of same type face-sequence also have links of the same face-sequence up to a cyclic permutation. A doubly semi-equivelar map M is called 2-uniform if it has 2 orbits of vertices under the action of its automorphism group. We denote the M of type $[f_1^{(f_{11},\ldots,f_{1r_1})} : f_2^{(f_{21},\ldots,f_{2r_2})}]$, where f_{1i} or f_{2j} is f_1 or f_2, for $1 \le i \le r_1$ and $1 \le j \le r_2$, if vertices of the face-sequence f_1 have links of face-sequence $(f_{11}, \ldots, f_{1r_1})$ and vertices of the face-sequence f_2 have links of face-sequence $(f_{21}, \ldots, f_{2r_2})$ respectively.

There are 20 types of 2-uniform tilings of the plane denoted as: $[3^6 : 3^3, 4^2]_1$, $[3^6 : 3^3, 4^2]_2$, $[3^6 : 3^2, 4, 3, 4]$, $[3^3, 4^2 : 3^2, 4, 3, 4]_1$, $[3^3, 4^2 : 3^2, 4, 3, 4]_2$, $[3^3, 4^2 : 4^4]_1$, $[3^3, 4^2 : 4^4]_2$, $[3^6 : 3^4, 6]_1$, $[3^6 : 3^4, 6]_2$, $[3^6 : 3^2, 4, 12]$, $[3^6 : 3^2, 6^2]$, $[3^4, 6 : 3^2, 6^2]$, $[3^3, 4^2 : 3, 4, 6, 4]_1$, $[3^2, 4, 3, 4, 4^2 : 3, 4, 6, 4]$, $[3^2, 6^2 : 3, 6, 3, 6]$, $[3, 4, 3, 12 : 3, 12^2]$, $[3, 4^2, 6 : 3, 4, 6, 4]$, $[3, 4^2, 6 : 3, 6, 3, 6]_1$, $[3, 4^2, 6 : 3, 6, 3, 6]_2$, $[3, 4, 6, 4 : 4, 6, 12]$, see [11]. Out of these, the first seven types have p-gons, with $p \le 4$, see Fig. 3.

2-uniform tilings of the plane

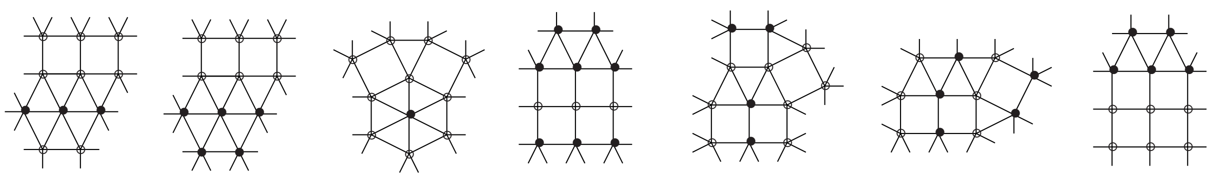


Fig. 3: 2-uniform tilings of types: $[3^6 : 3^3, 4^2]_1$, $[3^6 : 3^3, 4^2]_2$, $[3^6 : 3^2, 4, 3, 4]$, $[3^3, 4^2 : 4^4]_1$, $[3^3, 4^2 : 3^2, 4, 3, 4]_1$, $[3^3, 4^2 : 3^2, 4, 3, 4]_2$, $[3^3, 4^2 : 4^4]_2$ (see from left).

We classify the DSEMs on torus and Klein bottle corresponding to the above seven tilings. We abbreviate the types of DSEMs by the same notation as used for the respective tilings, see Table 3.

Table 3: Tabulated list of DSEMs of face-size 4

S No.	Abbreviated form	DSEM type
1.	$[3^6 : 3^3, 4^2]_1$	$[(3^6)^{((3^6),(3^3,4^2),(3^3,4^2),(3^6),(3^3,4^2),(3^3,4^2))}:$ $(3^3, 4^2)^{((3^6),(3^6),(3^3,4^2),(3^3,4^2),(3^3,4^2),(3^3,4^2),(3^3,4^2))}]$
2.	$[3^6 : 3^3, 4^2]_2$	$[(3^6)^{((3^6),(3^6),(3^6),(3^6),(3^3,4^2),(3^3,4^2))}:$ $(3^3, 4^2)^{((3^6),(3^6),(3^3,4^2),(3^3,4^2),(3^3,4^2),(3^3,4^2),(3^3,4^2))}]$
3.	$[3^6 : 3^2, 4, 3, 4]$	$[(3^6)^{((3^2,4,3,4),(3^2,4,3,4),(3^2,4,3,4),(3^2,4,3,4),(3^2,4,3,4),(3^2,4,3,4))}:$ $(3^2, 4, 3, 4)^{((3^6),(3^2,4,3,4),(3^2,4,3,4),(3^2,4,3,4),(3^2,4,3,4),(3^2,4,3,4),(3^2,4,3,4))}]$
4.	$[3^3, 4^2 : 3^2, 4, 3, 4]_1$	$[(3^3, 4^2)^{((3^3,4^2),(3^2,4,3,4),(3^2,4,3,4),(3^2,4,3,4),(3^2,4,3,4),(3^2,4,3,4),(3^2,4,3,4))}:$ $(3^2, 4, 3, 4)^{((3^3,4^2),(3^2,4,3,4),(3^2,4,3,4),(3^2,4,3,4),(3^3,4^2),(3^3,4^2),(3^2,4,3,4))}]$
5.	$[3^3, 4^2 : 3^2, 4, 3, 4]_2$	$[(3^3, 4^2)^{((3^3,4^2),(3^2,4,3,4),(3^2,4,3,4),(3^2,4,3,4),(3^3,4^2),(3^3,4,3,4),(3^3,4,3,4))}:$ $(3^2, 4, 3, 4)^{((3^3,4^2),(3^3,4^2),(3^3,4^2),(3^2,4,3,4),(3^3,4^2),(3^3,4^2),(3^2,4,3,4))}]$
6.	$[3^3, 4^2 : 4^4]_1$	$[(3^3, 4^2)^{((3^3,4^2),(3^3,4^2),(3^3,4^2),(3^3,4^2),(4^4),(4^4),(4^4))}:$ $(4^4)^{((3^3,4^2),(3^3,4^2),(3^3,4^2),(4^4),(3^3,4^2),(3^3,4^2),(3^3,4^2),(4^4))}]$
7.	$[3^3, 4^2 : 4^4]_2$	$[(3^3, 4^2)^{((3^3,4^2),(3^3,4^2),(3^3,4^2),(3^3,4^2),(4^4),(4^4),(4^4))}:$ $(4^4)^{((3^3,4^2),(3^3,4^2),(3^3,4^2),(4^4),(4^4),(4^4),(4^4),(4^4))}]$

3.1 Methodology

Each doubly semi-equivelar map, out of the seven types (listed in Table 3), contains two types of face-sequences among the four types (3^6), $(3^3, 4^2)$, $(3^2, 4, 3, 4)$ and (4^4) around the vertices. We use the following notations frequently to denote a vertex with specific type face-sequence in the computation. Here $\text{lk}(v)$ means link of vertex v.

- The notation $\text{lk}(v) = C_6(a, b, c, d, e, f)$ means the face-sequence of v is (3^6), i.e., the triangular faces $[v, b, c]$, $[v, c, d]$, $[v, d, e]$, $[v, e, f]$, $[v, f, a]$, $[v, b, a]$ incident at v.
- $\text{lk}(v) = C_7(a, b, [c, d, \boldsymbol{e}, f, g])$ means the face-sequence of v is $(3^3, 4^2)$, i.e., the triangular faces $[v, a, g]$, $[v, a, b]$, $[v, b, c]$ and quadrangular faces $[v, c, d, e]$, $[v, e, f, g]$ are incident at v.
- $\text{lk}(v) = C_7(a, [b, c, d], [e, f, g])$ means the face-sequence of v is $(3^2, 4, 3, 4)$, i.e., the triangular faces $[v, a, b]$, $[v, a, g]$, $[v, d, e]$ and quadrangular faces $[v, b, c, d]$, $[v, e, f, g]$ are incident at v.
- $\text{lk}(v) = C_8(\boldsymbol{a}, b, \boldsymbol{c}, d, \boldsymbol{e}, f, \mathbf{g}, h)$ means the face-sequence of v is (4^4), i.e., the quadrangular faces $[v, a, b, c]$, $[v, c, d, e]$, $[v, e, f, g]$ and $[v, g, h, a]$ are incident at v.

Since a doubly semi-equivelar map contains two types of vertices, in terms of face-sequences. Therefore to distinguish these vertices, we denote vertices of one type face-sequence by n and the other type by a_n, for some $n \in \mathbb{N}$. We describe a methodology to compute and classify the DSEMs listed in Table 3. Without loss of generality, we illustrate the methodology for type $[3^6 : 3^3.4^2]_1$. The same procedure is used for the remaining six types.

Let M be a DSEM of type $[3^6 : 3^3.4^2]_1$ with vertex set V on a surface of Euler characteristic 0 (i.e., on torus or Klein bottle). Let $V_{(3^6)}$ and $V_{(3^3,4^2)}$ denote the set of vertices with face-sequence type (3^6) and $(3^3, 4^2)$ respectively. Here $|V_{(3^6)}|$ and $|V_{(3^3,4^2)}|$ denote the cardinality of the sets $V_{(3^6)}$ and $V_{(3^3,4^2)}$ respectively. Then, it is easy to see that the number of triangular faces is $4|V_{(3^6)}|$ or $2|V_{(3^3,4^2)}|$. Thus, if the map exists then $2|V_{(3^6)}| = |V_{(3^3,4^2)}|$. Therefore we have $V = V_{(3^6)} \cup V_{(3^3,4^2)} = \{a_1, a_2, \ldots, a_{|V_{(3^6)}|}, 1, 2, \ldots, |V_{(3^3,4^2)}|\}$ such that $2|V_{(3^6)}| = |V_{(3^3,4^2)}|$. Now we use the following steps to enumerate DSEM M for this $V = V_{(3^6)} \cup V_{(3^3,4^2)}$.

Steps to enumerate DSEMs of type $[3^6 : 3^3.4^2]_1$:

Step 1:

1. Without loss of generality, let us start with a vertex v_1 having face-sequence type (3^6). Let $\text{lk}(a_1) = C_6(a_2, 1, 2, a_3, 3, 4)$.
2. This implies $\text{lk}(a_2) = C_6(1, a_1, 4, n_2, x_1, n_1)$ with several choices for the triplet (n_1, x_1, n_2) in $V_{(3^3,4^2)} \times V_{(3^6)} \times V_{(3^3,4^2)}$ or $\text{lk}(4) = C_7(a_1, a_2, [n_1, n_2, \boldsymbol{n}_3, n_4, 3])$ with several choices for $(n_1, n_2, n_3, n_4) \in V_{(3^3,4^2)} \times V_{(3^3,4^2)} \times V_{(3^3,4^2)} \times V_{(3^3,4^2)}$, see Fig. 3.1.
3. Again among $\text{lk}(a_2)$ and $\text{lk}(4)$, without loss of generality, we proceed with $\text{lk}(a_2) = C_6(1, a_1, 4, n_2, x_1, n_1)$. For each choice of (n_1, x_1, n_2) we have distinct possibility for $\text{lk}(a_2)$. Out of these possibilities of $\text{lk}(a_2)$, we qualify those ones which preserves the face-sequence types of vertices. The similar procedure may be adopted for $\text{lk}(4)$ (if required).

Step 2: We continuously repeat Step 1 until we do not get the links of remaining vertices from V.

Step 3: The computation involves in Step 1 and Step 2 is case by case and exhaustive covering all possible scenarios.

Step 4: We explore isomorphism between the maps obtained in Step 1 and Step 2, which leads to the enumeration of DSEMs of type $[3^6 : 3^3.4^2]_1$.

To show that two maps M_1 and M_2 are non-isomorphic, we compute the characteristic polynomials $p(EG(M_1))$ and $p(EG(M_2))$ of adjacency matrices associated the edge graphs $EG(M_1)$ and $EG(M_2)$ of the maps M_1 and M_2 respectively. The edge graph of M is a graph $EG(M)$ consisting of vertices and edges of the map. Clearly if $p(EG(M_1)) \neq p(EG(M_2))$, $M_1 \ncong M_2$. However if $p(EG(M_1)) = p(EG(M_2))$, we can not say anything.

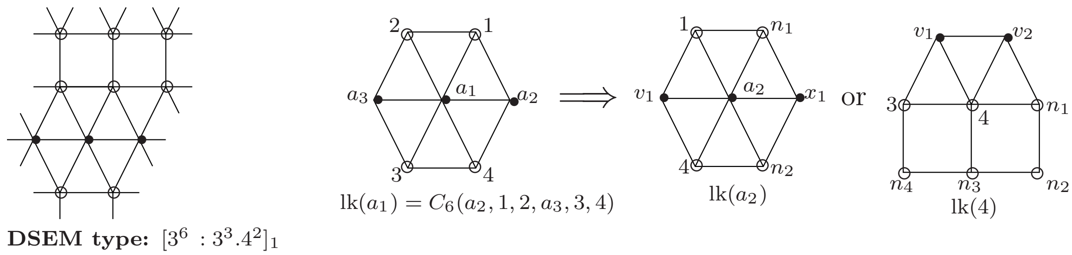


Fig. 3.1: Illustration of the methodology

4 Computation and classification of DSEMs

In this section, we compute and classify the seven types DSEMs (listed in Table 3) using the methodology given in Sec. 3. For the sake of computation, we consider the number of vertices ≤ 15.

4.1 Computation and classification for type $[3^6\text{:}\ 3^3, 4^2]_1$

Consider the following DSEMs of type $[3^6 : 3^3, 4^2]_1$, in Fig 4.1, on torus and Klein bottle denoted by $T_{i(n,2n)}[3^6 : 3^3, 4^2]_1$, for $i \in \{1, \ldots, 8\}$, and $K_{i(n,2n)}[3^6 : 3^3, 4^2]_1$, for $i \in \{1, \ldots, 6\}$, respectively.

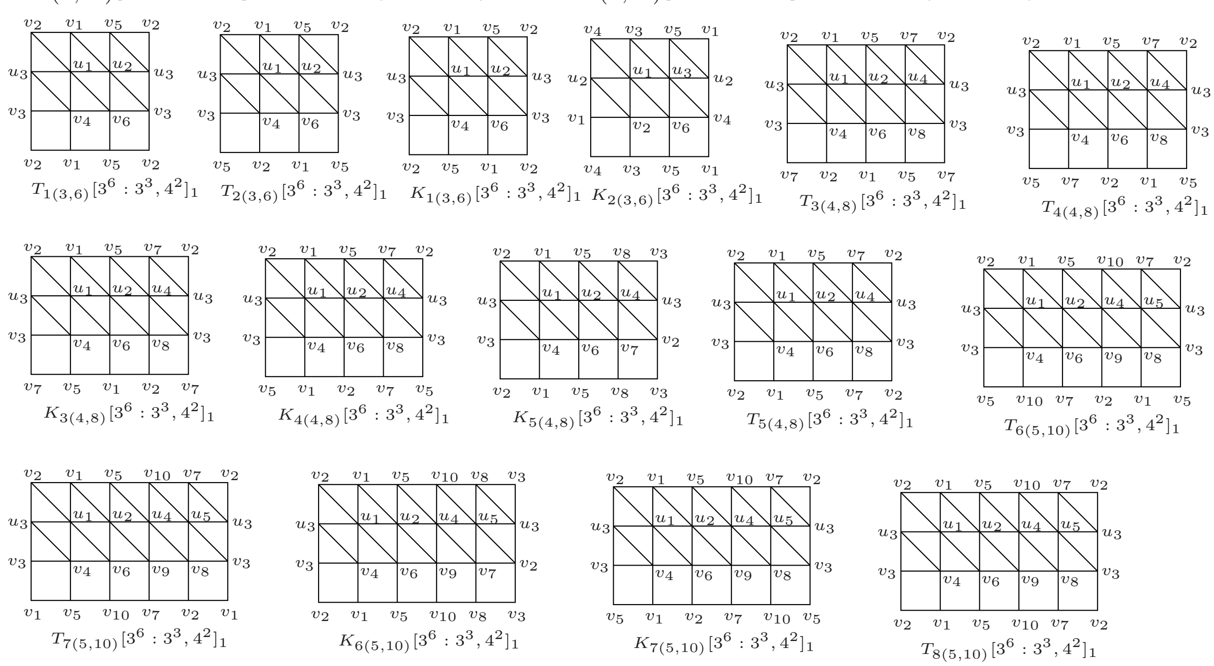


Fig. 4.1: Doubley semi-equivelar maps on torus and Klein bottle of type $[3^6 : 3^3, 4^2]_1$

Claim 4.1 *For the maps above, we have the following:*

(*a*) $T_{1(3,6)}[3^6 : 3^3, 4^2]_1 \ncong T_{2(3,6)}[3^6 : 3^3, 4^2]_1$.

(*b*) $T_{3(4,8)}[3^6 : 3^3, 4^2]_1 \ncong T_{4(4,8)}[3^6 : 3^3, 4^2]_1 \ncong T_{5(4,8)}[3^6 : 3^3, 4^2]_1$.

(*c*) $K_{3(4,8)}[3^6 : 3^3, 4^2]_1 \ncong K_{4(4,8)}[3^6 : 3^3, 4^2]_1 \ncong K_{5(4,8)}[3^6 : 3^3, 4^2]_1$.

(d) $T_{6(5,10)}[3^6:3^3,4^2]_1 \ncong T_{7(5,10)}[3^6:3^3,4^2]_1 \ncong T_{8(5,10)}[3^6:3^3,4^2]_1$.

(e) $K_{6(5,10)}[3^6:3^3,4^2]_1 \ncong K_{7(5,10)}[3^6:3^3,4^2]_1$.

Proof. Let $p(EG(M))$ denote the characteristic polynomial of adjacency matrix associated with the edge graph of M. Then the proof follows from the following polynomials:

$p(EG(T_{1(3,6)}[3^6:3^3,4^2]_1)) = x^9-24x^7-42x^6+63x^5+138x^4-72x^3-144x^2+48x+32$,

$p(EG(T_{2(3,6)}[3^6:3^3,4^2]_1)) = x^9-24x^7-36x^6+45x^5+48x^4-21x^3-18x^2+3x+2$,

$p(EG(T_{3(4,8)}[3^6:3^3,4^2]_1)) = x^{12}-32x^{10}-40x_9+254x^8+440x^7-628x^6-1400x^5+105x^4+1000x^3+300x^2$,

$p(EG(T_{4(4,8)}[3^6:3^3,4^2]_1)) = x^{12}-32x^{10}-32x_9+254x^8+224x^7-932x^6-448x^5+1673x^4+96x^3-1156x^2+160x+192$,

$p(EG(T_{5(4,8)}[3^6:3^3,4^2]_1)) = x^{12}-32x^{10}-48x_9+254x^8+656x^7-292x^6-2352x^5-2167x^4+624x^3+2044x^2+1120x+192$,

$p(EG(K_{3(4,8)}[3^6:3^3,4^2]_1)) = x^{12}-31x^{10}-39x_9+227x^8+377x^7-561x^6-1129x^5+416x^4+1283x^3+92x^2-492x-144$,

$p(EG(K_{4(4,8)}[3^6:3^3,4^2]_1)) = x^{12}-32x^{10}-40x_9+254x^8+440x^7-644x^6-1400x^5+457x^4+1640x^3+156x^2-640x-192$,

$p(EG(K_{5(4,8)}[3^6:3^3,4^2]_1)) = x^{12}-32x^{10}-48x_9+258x^8+640x^7-364x^6-220x^5-1635x^4+496x^3+684x^2-32x-64$,

$p(EG(T_{6(5,10)}[3^6:3^3,4^2]_1)) = x^{15}-40x^{13}-40x^{12}+515x^{11}+754x^{10}-282x_9-4940x^8+6790x^7+13430x^6-668x^5-15340x^4+975x^3+5490x^2+1755x+162$,

$p(EG(T_{7(5,10)}[3^6:3^3,4^2]_1)) = x^{15}-38x^{13}-51x^{12}+462x^{11}+1033x^{10}-1049x_9-5533x^8-4681x^7+2905x^6+6351x^5+2282x^4-1046x^3-680x^2+32x+48$,

$p(EG(T_{8(5,10)}[3^6:3^3,4^2]_1)) = x^{15}-40x^{13}-60x^{12}+485x^{11}+1374x^{10}-985x_9-7910x^8-9955x^7-1010x^6+7623x^5+7030x^4+2820x^3+570x^2+55x+2$,

$p(EG(K_{6(5,10)}[3^6:3^3,4^2]_1)) = x^{15}-39x^{13}-58x^{12}+462x^{11}+1309x^{10}-916x_9-7455x^8-9096x^7-203x^6+6562x^5+3147x^4-909x^3-761x^2-97x-3$,

$p(EG(K_{7(5,10)}[3^6:3^3,4^2]_1)) = x^{15}-40x^{13}-48x^{12}+497x^{11}+1010x^{10}-1973x_9-6234x^8-111x^7+12010x^6+9531x^5-3294x^4-7264x^3-3410x^2-637x-38$.

Claim 4.2 $K_{1(3,6)}[3^6:3^3,4^2]_1 \ncong K_{2(3,6)}[3^6:3^3,4^2]_1$.

proof. Note that $p(EG(K_{1(3,6)}[3^6:3^3,4^2]_1)) = p(EG(K_{2(3,6)}[3^6:3^3,4^2]_1)) = x^9-24x^7-38x^6+51x^5+78x^4-44x^3-24x^2$, but the maps are non-isomorphic. To see this, we use geometric argument as follows: Define a basis $\{a,b\}$ at any vertex v_i (for $1 \le i \le 6$), where a and b are minimal non-trivial loops (*i.e.* non-trivial cycle with minimum number of vertices), now if we consider $K_{1(3,6)}[3^6:3^3,4^2]_1$ then at each v_i, we get a and b with length 3 (for example at v_1, $a=C_3(v_1,v_6,u_2)$ and $b=C_3(v_1,v_2,v_5)$) while in $K_{2(3,6)}[3^6:3^3,4^2]_1$, at each v_i, we get a of length 3 and b of length 4 (for example at v_1, we see that $a=C_3(v_4,v_1,u_1)$ or $a=C_3(v_4,v_1,u_2)$ and $b=C_4(v_1,v_2,v_6,v_4)$). Hence, $K_{1(3,6)}[3^6:3^3,4^2]_1 \ncong K_{2(3,6)}[3^6:3^3,4^2]_1$.

4.1.1 Computation:

Let M be a DSEM of type $[3^6\colon 3^3,4^2]_1$ with the vertex set V. Let $V_{(3^6)}$ and $V_{(3^3,4^2)}$ denote the sets of vertices with face-sequence types (3^6) and $(3^3,4^2)$, respectively. Then, we see that the number of triangular faces in M is $4|V_{(3^6)}|$ or $2|V_{(3^3,4^2)}|$. This implies $2|V_{(3^6)}| = |V_{(3^3,4^2)}|$. Thus for $|V| = (|V_{(3^3,4^2)}|+|V_{(3^3,4^2)}|) \le 15$, we let $V=\{a_1,a_2,\ldots,a_{|V_{(3^6)}|},1,2,\ldots,2|V_{(3^6)}|\}$, where $|V_{(3^6)}| \le 5$. Without loss of generality, we may assume $\mathrm{lk}(a_1)=C_6(a_2,1,2,a_3,3,4)$. This implies $\mathrm{lk}(a_2)=C_6(a_1,1,n_1,x_1,n_2,4)$, $\mathrm{lk}(1)=C_7(a_1,a_2,[n_1,n_3,\boldsymbol{n}_4,n_5,2])$, $\mathrm{lk}(2)=C_7(a_1,a_3,[n_6,n_7,\boldsymbol{n}_5,n_4,1])$, $\mathrm{lk}(a_3)=C_6(a_1,3,n_8,x_2,n_6,2)$, $\mathrm{lk}(3)=C_7(a_1,a_3,[n_8,n_9,\boldsymbol{n}_{10},n_{11},4])$ and $\mathrm{lk}(4)=C_7(a_1,a_2,[n_2,n_{12},\boldsymbol{n}_{11},n_{10},3])$ for some $x_1,x_2 \in V_{(3^6)}$ and $n_1,n_2,\ldots,n_{12} \in V_{(3^3,4^2)}$.

Now considering $\mathrm{lk}(a_2)$, we see that $n_1 \ne 2$ or 3 (for $n_1=2$, the set $\{2,1,a_2\}$ forms triangular face in $\mathrm{lk}(a_1)$ but not in $\mathrm{lk}(a_2)$, for $n_1=3$ we get $\deg(3)>5$). Similarly we see that $n_2 \notin \{2,3\}$. From these observations, we have $(n_1,x_1,n_2) \in \{(5,a_3,6),(5,a_4,6)\}$.

Case 1: If $(n_1, x_1, n_2) = (5, a_3, 6)$, then $\text{lk}(a_3) = C_6(a_1, 3, 6, a_2, 5, 2)$ or $\text{lk}(a_3) = C_6(a_1, 3, 5, a_2, 6, 2)$.

When $\text{lk}(a_3) = C_6(a_1, 3, 5, a_2, 6, 2)$, then considering lk(2) we have $(n_4, n_5, n_7) \in \{(4, 3, 5), (5, 3, 4)\}$. If $(n_4, n_5, n_7) = (5, 3, 4)$, then lk(1) is a cycle of length 5, a contradiction. On the other hand, if $(n_4, n_5, n_7) = (4, 3, 5)$ then $\text{lk}(2) = C_7(a_1, a_3, [6, 5, \mathbf{3}, 4, 1])$, $\text{lk}(1) = C_7(a_1, a_2, [5, 6, \mathbf{4}, 3, 2])$, completing successively, we get $\text{lk}(4) = C_7(a_1, a_2, [6, 5, \mathbf{1}, 2, 3])$, $\text{lk}(3) = C_7(a_1, a_3, [5, 6, \mathbf{2}, 1, 4])$, $\text{lk}(5) = C_7(a_2, a_3, [3, 2, \mathbf{6}, 4, 1])$ and $\text{lk}(6) = C_7(a_2, a_3, [2, 3, \mathbf{5}, 1, 4])$. Then we get $M \cong K_{2(3,6)}[3^6\colon 3^3, 4^2]_1$ by the map $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 6$, $1 \le j \le 3$.

When $\text{lk}(a_3) = C_6(a_1, 3, 6, a_2, 5, 2)$, considering lk(2), we get $(n_4, n_5, n_7) \in \{(4, 3, 6), (6, 3, 4), (3, 4, 6), (6, 4, 3), (3, 6, 4), (4, 6, 3)\}$. Observe that, $(3, 4, 6) \cong (6, 3, 4)$ by the map $(1, 5, 2)(3, 4, 6)(a_1, a_2, a_3)$, $(3, 6, 4) \cong (6, 4, 3)$ by the map $(1, 5)(4, 6)(a_1, a_3)$ and $(4, 6, 3) \cong (6, 3, 4)$ by the map $(1, 2, 5)(3, 6, 4)(a_1, a_3, a_2)$. So, we need to search only for $(n_4, n_5, n_7) \in \{(4, 3, 6), (6, 3, 4), (6, 4, 3)\}$.

In case $(n_4, n_5, n_7) = (4, 3, 6)$, completing successively, we get $\text{lk}(2) = C_7(a_1, a_3, [5, 6, \mathbf{3}, 4, 1])$, $\text{lk}(3) = C_7(a_1, a_3, [6, 5, \mathbf{2}, 1, 4])$, $\text{lk}(4) = C_7(a_1, a_2, [6, 5, \mathbf{1}, 2, 3])$, $\text{lk}(1) = C_7(a_1, a_2, [5, 6, \mathbf{4}, 3, 2])$, $\text{lk}(5) = C_7(a_2, a_3, [2, 3, \mathbf{6}, 4, 1])$, $\text{lk}(6) = C_7(a_2, a_3, [3, 2, \mathbf{5}, 1, 4])$. This gives $M \cong T_{1(3,6)}[3^6\colon 3^3, 4^2]_1$ by the map $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 6$, $1 \le j \le 3$.

Proceeding similarly as above, for $(n_4, n_5, n_7) = (6, 3, 4)$ we get $M \cong K_{1(3,6)}(3^6 : 3^3, 4^2)_1$ by the map $i \mapsto v_i$, $a_j \mapsto u_j$, where $1 \le i \le 6$, $1 \le j \le 3$.

For $(n_4, n_5, n_7) = (6, 4, 3)$, $M \cong T_{2(3,6)}(3^6 : 3^3, 4^2)_1$, by the map $i \mapsto v_i$, $a_j \mapsto u_j$, for $1 \le i \le 6$, $1 \le j \le 3$.

Case 2. For $(n_1, x_1, n_2) = (5, a_4, 6)$ considering $\text{lk}(a_3)$ we get $x_2 \in \{a_4, a_5\}$.

Subcase 2.1. If $x_2 = a_4$, then $\text{lk}(a_3) = C_6(a_1, 2, 7, a_4, 8, 3)$. This implies $\text{lk}(a_4) = C_6(a_2, 5, 7, a_3, 8, 6)$ or $\text{lk}(a_4) = C_6(a_2, 5, 8, a_3, 7, 6)$.

When $\text{lk}(a_4) = C_6(a_2, 5, 7, a_3, 8, 6)$, then, up to isomorphism, we see that $(n_{13}, n_{14}, n_{15}) \in \{(2, 1, 5), (5, 1, 2), (1, 2, 7), (7, 2, 1), (7, 5, 1)\}$. Now doing computation for these case, we see:

If $(n_{13}, n_{14}, n_{15}) = (2, 1, 5)$, $M \cong K_{3(4,8)}[3^6\colon 3^3, 4^2]_1$ by $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 8$, $1 \le j \le 4$.

If $(n_{13}, n_{14}, n_{15}) = (5, 1, 2)$, $M \cong T_{3(4,8)}[3^6\colon 3^3, 4^2]_1$ by $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 8$, $1 \le j \le 4$.

If $(n_{13}, n_{14}, n_{15}) = (1, 2, 7)$, $M \cong T_{4(4,8)}[3^6 : 3^3, 4^2]_1$ by $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 8$, $1 \le j \le 4$.

If $(n_{13}, n_{14}, n_{15}) = (7, 2, 1)$, $M \cong K_{4(4,8)}[3^6 : 3^3, 4^2]_1$ by $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 8$, $1 \le j \le 4$.

If $(n_{13}, n_{14}, n_{15}) = (7, 5, 1)$, $M \cong T_{5(4,8)}[3^6 : 3^3, 4^2]_1$ by $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 8$, $1 \le j \le 4$.

On the other hand when $\text{lk}(a_4) = C_6(a_2, 5, 8, a_3, 7, 6)$, we get $(n_{13}, n_{14}, n_{15}) \in \{(2, 1, 5), (1, 5, 8), (3, 8, 5), (5, 1, 2), (5, 8, 3), (8, 5, 1)\}$. If $(n_{13}, n_{14}, n_{15}) = (2, 1, 5)$ and $(1, 5, 8)$, then lk(7) is a cycle of length 5 and 6 respectively, which is not possible. If $(n_{13}, n_{14}, n_{15}) = (3, 8, 5)$ and $(5, 1, 2)$, then we see easily that lk(7) and lk(8) can not be completed respectively. If $(n_{13}, n_{14}, n_{15}) = (5, 8, 3)$, then completing lk(7) we get lk(1) of length 5, again a contradiction. If $(n_{13}, n_{14}, n_{15}) = (8, 5, 1)$, $M \cong K_{5(4,8)}[3^6 : 3^3, 4^2]_1$ by $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 8$, $1 \le j \le 4$.

Subcase 2.2. When $x_2 = a_5$, successively, we get $\text{lk}(a_3) = C_6(a_1, 2, 7, a_5, 8, 3)$ and $\text{lk}(a_4) = C_6(a_2, 6, 9, a_5, 10, 5)$. This implies $\text{lk}(a_5) = C_6(a_4, 9, 7, a_3, 8, 10)$ or $\text{lk}(a_5) = C_6(a_4, 9, 8, a_3, 7, 10)$.

In case $\text{lk}(a_5) = C_6(a_4, 9, 7, a_3, 8, 10)$, considering lk(1), we get $(n_3, n_4, n_5) \in \{(3, 4, 6), (3, 8, 10), (4, 3, 8), (4, 6, 9), (6, 4, 3), (6, 9, 7), (7, 9, 6), (8, 3, 4), (9, 6, 4), (10, 8, 3)\}$. But a small calculation shows that no map exists for these cases, except for $(n_3, n_4, n_5) = (6, 4, 3)$. For $(n_3, n_4, n_5) = (6, 4, 3)$ we get $M \cong K_{6(5,10)}[3^6 : 3^3, 4^2]_1$ by the map $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 10$, $1 \le j \le 5$.

On the other hand when $\text{lk}(a_5) = C_6(a_4, 9, 8, a_3, 7, 10)$, considering lk(1), up to isomorphism, we get $(n_3, n_4, n_5) \in \{(3, 4, 6), (3, 8, 9), (4, 3, 8), (6, 4, 3)\}$. Now doing computation for these cases, we see:

If $(n_3, n_4, n_5) = (3, 4, 6)$, $M \cong K_{7(5,10)}[3^6 : 3^3, 4^2]_1$ by $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 10$ and $1 \le j \le 5$.

If $(n_3, n_4, n_5) = (3, 8, 9)$, $M \cong T_{6(5,10)}[3^6 : 3^3, 4^2]_1$ by $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 10$ and $1 \le j \le 5$.

If $(n_3, n_4, n_5) = (4, 3, 8)$, $M \cong T_{7(5,10)}[3^6 : 3^3, 4^2]_1$ by $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 10$ and $1 \le j \le 5$.

If $(n_3, n_4, n_5) = (6, 4, 3)$, $M \cong T_{8(5,10)}[3^6 : 3^3, 4^2]_1$ by $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 10$ and $1 \le j \le 5$.

This completes computation for the number of vertices ≤ 15 and we obtain the following results.

4.1.2 Results

Lemma 4.1 *Let M be a DSEM of type $[3^6 : 3^3, 4^2]_1$ with number of vertices ≤ 15. Then M is isomorphic to one of the following: $T_{1(3,6)}[3^6 : 3^3, 4^2]_1$, $T_{2(3,6)}[3^6 : 3^3, 4^2]_1$, $K_{1(3,6)}[3^6 : 3^3, 4^2]_1$, $K_{2(3,6)}[3^6 : 3^3, 4^2]_1$, $T_{3(4,8)}[3^6 : 3^3, 4^2]_1$, $T_{4(4,8)}[3^6 : 3^3, 4^2]_1$, $T_{5(4,8)}[3^6 : 3^3, 4^2]_1$, $K_{3(4,8)}[3^6 : 3^3, 4^2]_1$, $K_{4(4,8)}[3^6 : 3^3, 4^2]_1$, $K_{5(4,8)}[3^6 : 3^3, 4^2]_1$, $T_{6(5,10)}[3^6 : 3^3, 4^2]_1$, $T_{7(5,10)}[3^6 : 3^3, 4^2]_1$, $T_{8(5,10)}[3^6 : 3^3, 4^2]_1$, $K_{6(5,10)}[3^6 : 3^3, 4^2]_1$, $K_{7(5,10)}[3^6 : 3^3, 4^2]_1$, shown in Fig. 4.1.*

Combining the Lemma 4.1 together with the Claims 4.1 and 4.2, it follows that:

Theorem 4.1 *There are exactly 15 DSEMs of type $[3^6 : 3^3, 4^2]_1$ on the surfaces of Euler characteristic 0 with number of vertices ≤ 15. Out of these 8 are on the torus and remaining 7 are on the Klein bottle.*

4.2 Computation and classification for type $[3^6\colon 3^3, 4^2]_2$

Consider the following DSEMs of type $[3^6\colon 3^3, 4^2]_2$, shown in Fig. 4.2, on torus and Klein bottle denoted by $T_{i(6,6)}[3^6 : 3^3, 4^2]_2$, for $i = 1, 2$, and $K_{1(6,6)}[3^6 : 3^3, 4^2]_2$, respectively.

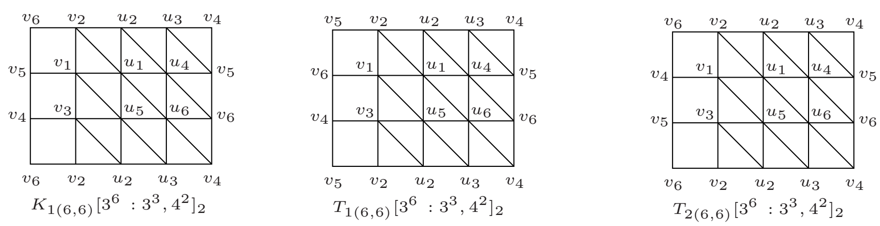


Fig. 4.2: Doubley semi-equivelar maps on torus and Klein bottle of type $[3^6 : 3^3, 4^2]_2$

Claim 4.3 $T_{1(6,6)}[3^6 : 3^3, 4^2]_2 \ncong T_{2(6,6)}[3^6 : 3^3, 4^2]_2$.

Proof. Follows from the following polynomials:

$p(EG(T_{1(6,6)}[3^6 : 3^3, 4^2]_2)) = x^{12} - 33x^{10} - 44x^9 + 258x^8 + 432x^7 - 682x^6 - 1032x^5 + 957x^4 + 560x^3 - 789x^2 + 276x - 32,$

$p(EG(T_{2(6,6)}[3^6 : 3^3, 4^2]_2)) = x^{12} - 33x^{10} - 44x^9 + 252x^8 + 456x^7 - 568x^6 - 1296x^5 + 348x^4 + 1328x^3 + 108x^2 - 432x - 128.$

4.2.1 Computation

Let M be a map of the type $[3^6 : 3^3, 4^2]_2$ with the vertex set V. Let V_{3^6} and $V_{3^3,4^2}$ denote the sets of vertices with face-sequence types (3^6) and $(3^3, 4^2)$, respectively. Observe that, M has the number of edges $= (5|V_{(3^6)}| + \frac{|V_{(3^3,4^2)}|}{2})$, number of triangular faces $= 3|V_{(3^6)}|$ and number of quadrangular faces $= \frac{|V_{(3^3,4^2)}|}{2}$. Now by the Euler characteristic equation, we get $(|V_{(3^6)}| + |V_{(3^3,4^2)}|) - (5|V_{(3^6)}| + \frac{|V_{(3^3,4^2)}|}{2}) + (3|V_{(3^6)}| + \frac{|V_{(3^3,4^2)}|}{2}) = 0$. This implies $|V_{(3^6)}| = |V_{(3^3,4^2)}|$. Also, considering the number of quadrangular faces, it is evident that the cardinality of both the sets should be positive even integer. Thus for $|V| \leq 15$, we let $V = \{a_1, a_2, \ldots, a_{|V_{(3^6)}|}, 1, 2, \ldots, |V_{(3^6)}|\}$, where $|V_{(3^6)}| = 2k$ for $k \leq 3$.

Without loss of generality, assume $\mathrm{lk}(a_1) = C_6(a_2, a_3, a_4, a_5, 1, 2)$. This implies $\mathrm{lk}(1) = C_7(a_1, a_5, [n_1, n_2, \boldsymbol{n}_3, n_4, 2])$, $\mathrm{lk}(2) = C_7(a_2, a_1, [1, n_3, \boldsymbol{n}_4, n_6, n_5])$ and $\mathrm{lk}(a_2) = C_6(a_1, a_3, x_1, x_2, 3, 2)$ for some $n_1, \ldots, n_6 \in V_m$ and $x_1, x_2 \in V_l$. It is easy to see that $(x_1, x_2) \in \{(a_5, a_4), (a_6, a_5)\}$.

Case 1. When $(x_1, x_2) = (a_5, a_4)$. Then, successively, we get $\mathrm{lk}(a_2) = C_6(a_1, a_3, a_5, a_4, 3, 2)$, $\mathrm{lk}(a_5) = C_6(a_1, a_4, a_2, a_3, 4, 1)$, $\mathrm{lk}(a_3) = C_6(a_4, a_1, a_2, a_5, 4, 5)$ and $\mathrm{lk}(a_4) = C_6(a_3, a_1, a_5, a_2, 3, 5)$. Now considering $\mathrm{lk}(1)$, we see that (n_2, n_3, n_4) has no value for the V so that the links of remaining vertices can be completed. So $(x_1, x_2) \neq (a_5, a_4)$.

Case 2. When $(x_1, x_2) = (a_6, a_5)$, then successively we get $\mathrm{lk}(a_2) = C_6(a_1, a_3, a_6, a_5, 3, 2)$, $\mathrm{lk}(a_5) = C_6(a_1, a_4, a_6, a_2, 3, 1)$, $\mathrm{lk}(a_4) = C_6(a_3, a_1, a_5, a_6, 5, 4)$, $\mathrm{lk}(a_3) = C_6(a_4, a_1, a_2, a_6, 6, 4)$, $\mathrm{lk}(a_6) = C_6(a_3, a_2, a_5, a_4, 5, 6)$. Considering lk(1), it is easy to see that, $(n_2, n_3, n_4) \in \{(4,5,6), (4,6,5), (5,4,6), (5,6,4), (6,4,5), (6,5,4)\}$.

Observe that $(4,5,6) \cong (6,4,5)$ by the map $(1,3,2)(5,6,4)(a_1, a_5, a_2)(a_3, a_4, a_6)$; $(4,5,6) \cong (5,6,4)$ by the map $(1,2,3)(4, 6,5)(a_1, a_2, a_5)(a_3, a_6, a_4)$ and $(5,4,6) \cong (6,5,4)$ by the map $(1,3)(4,6)(a_1, a_2)(a_4, a_6)$. So we search for $(n_2, n_3, n_4) \in \{(4,5,6), (4,6,5), (5,4,6)\}$. Now doing computation for these cases, we see:

If $(n_2, n_3, n_4) = (4,5,6)$, $M \cong K_{1(6,6)}[3^6 : 3^3, 4^2]_2$ by the map $i \mapsto v_i$, $a_i \mapsto u_i$, $1 \le i \le 6$.
If $(n_2, n_3, n_4) = (4,6,5)$, $M \cong T_{1(6,6)}[3^6 : 3^3, 4^2]_2$ by the map $i \mapsto v_i$, $a_i \mapsto u_i$, $1 \le i \le 6$
If $(n_2, n_3, n_4) = (5,4,6)$, $M \cong T_{2(6,6)}[3^6 : 3^3, 4^2]_2$ by the map $i \mapsto v_i$, $a_i \mapsto u_i$, $1 \le i \le 6$.
This completes computation for ≤ 15 vertices. From this we get following results.

4.2.2 Results

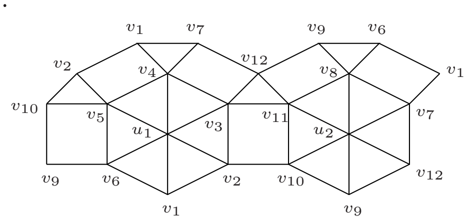
Lemma 4.2 *Let M be a DSEM of type $[3^6 : 3^3, 4^2]_2$ on the surfaces of Euler characteristic 0 with ≤ 15 vertices. Then M is isomorphic to one of $K_{1(6,6)}[3^6 : 3^3, 4^2]_2$, $T_{1(6,6)}[3^6 : 3^3, 4^2]_2$ and $T_{1(6,6)}[3^6 : 3^3, 4^2]_2$, given in Fig. 4.2.*

Combining Lemma 4.2 with Claim 4.3, it follows that:

Theorem 4.2 *There are exactly 3 non-isomorphic DSEMs of type $[3^6 : 3^3, 4^2]_2$ with number of vertices ≤ 15. Out of these 2 are on torus and remaining one on Klein bottle.*

4.3 Computation and classification for type $[3^6 : 3^2, 4, 3, 4]$

Consider the following DSEM of type $[3^6 : 3^2, 4, 3, 4]$, given in Fig. 4.3 on Klein bottle denoted by $K_{1(2,12)}[3^6 : 3^2, 4, 3, 4]$.



$K_{1(2,12)}[3^6 : 3^2, 4, 3, 4]$

Fig. 4.3: Doubley semi-equivelar maps on Klein bottle of type $[3^6 : 3^2, 4, 3, 4]$

4.3.1 Computation

Let M be a map of the type $[3^6 : 3^2, 4, 3, 4]$ with the vertex set V. Let $V_{(3^6)}$ and $V_{(3^2,4,3,4)}$ denote the sets of vertices with face-sequence types (3^6) and $(3^2, 4, 3, 4)$ respectively. It is easy to see that $6|V_{(3^6)}| = |V_{(3^2,4,3,4)}|$. Thus, for $|V| \le 15$, we let $V = \{a_1, a_2, \ldots, a_{|V_{(3^6)}|}, 1, 2, \ldots, 6|V_{(3^6)}|\}$, where $|V_{(3^6)}| \le 2$. Without loss of generality, we assume $\mathrm{lk}(a_1) = C_6(1, 2, 3, 4, 5, 6)$. Then, successively, we have $\mathrm{lk}(1) = C_7(a_1, [2, n_1, n_2], [n_3, n_4, 6])$, $\mathrm{lk}(2) = C_7(a_1, [1, n_2, n_1], [n_5, n_6, 3])$, $\mathrm{lk}(3) = C_7(a_1, [2, n_5, n_6], [n_7, n_8, 4])$, $\mathrm{lk}(4) = C_7(a_1, [3, n_7, n_8], [n_9, n_{10}, 5])$, $\mathrm{lk}(5) = C_7(a_1, [4, n_9, n_{10}], [n_{11}, n_{12}, 6])$, $\mathrm{lk}(6) = C_7(a_1, [1, n_3, n_4], [n_{12}, n_{11}, 5])$, where $n_i \in V_m$ for $1 \le i \le 12$. Now considering lk(1), we see that $n_1 \in \{4, 5, 7\}$.
Case 1: If $n_1 = 4$, then successively, we see that $n_2 = 5$, $n_3 = 7$, $n_4 = 8$, now considering lk(5) and lk(1), we get two quadrangular faces which share more than one vertex, which is not allowed. So $n_1 \ne 4$.
Case 2: If $n_1 = 5$, then successively, we get $n_2 = 4$, $n_3 = 7$, $n_4 = 8$, $n_{12} = 9$, $n_{11} = 10$. Now completing lk(1), lk(6), lk(5), lk(2), lk(3), lk(4), lk(7) and lk(10) we see that, $\mathrm{lk}(a_2) =$

$C_6(7,8,11,10,9,12)$ or $\text{lk}(a_2) = C_6(7,8,9,10,11,12)$. If $\text{lk}(a_2) = C_6(7,8,9,10,11,12)$, then lk(8) is a cycle of length 5, a contradiction. If $\text{lk}(a_2) = C_6(7,8,11,10,9,12)$, then completing successively, we get $M \cong K_{1(2,12)}[3^6 : 3^2,4,3,4]$ by the map $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 12$, $1 \le j \le 2$.

Case 3: If $n_1 = 7$, then we get $(n_2, n_3) \in \{(8,3),(8,4),(8,9)\}$.

For $(n_2,n_3) = (8,3)$, $n_4 = 4$ and $\text{lk}(1) = C_7(a_1,[2,7,8],[3,4,6])$. Now considering successively lk(3) and lk(1) we see two quadrangular faces which share more than one vertex, which is not allowed. Hence $(n_2,n_3) \neq (8,3)$.

For $(n_2,n_3) = (8,4)$, completing successively lk(1), lk(4), lk(5), lk(6), lk(3), lk(2), lk(11), lk(8) we see that, $\text{lk}(a_2) = C_6(7,8,9,10,11,12)$ or $\text{lk}(a_2) = C_6(7,8,9,12,11,10)$.

If $\text{lk}(a_2) = C_6(7,8,9,10,11,12)$, then lk(7) is a cycle of length 5, a contradiction.

If $\text{lk}(a_2) = C_6(7,8,9,12,11,10)$, completing successively, we get $M \cong K_{1(2,12)}$-$[3^6 : 3^2,4,3,4]$ via $1 \mapsto v_9$, $2 \mapsto v_{10}$, $3 \mapsto v_{11}$, $4 \mapsto v_8$, $5 \mapsto v_7$, $6 \mapsto v_{12}$, $7 \mapsto v_5$, $8 \mapsto v_6$, $9 \mapsto v_1$, $10 \mapsto v_4$, $11 \mapsto v_3$, $12 \mapsto v_2$, $a_1 \mapsto u_2$, $a_2 \mapsto u_1$.

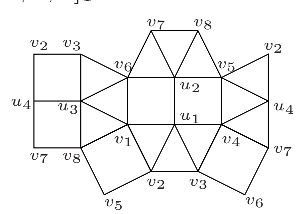
If $(n_2,n_3) = (8,9), n_4 = 10$. This implies $\text{lk}(1) = C_7(a_1,[2,7,8],[9,10,6])$, $\text{lk}(6) = C_7(a_1,[1,9,10],[n_{12},n_{11},5])$ for $(n_{11},n_{12}) \in \{(3,2),(11,4),(11,7),(11,12)\}$. A small calculation shows, no map exists for $(n_{11},n_{12}) \in \{(11,4),(11,7),\ (11,12)\}$. For $(n_{11},n_{12}) = (3,2)$, completing successively, we get $M \cong K_{1(2,12)}[3^6 : 3^2,4,3,4]$ via $1 \mapsto v_3$, $2 \mapsto v_4$, $3 \mapsto v_5$, $4 \mapsto v_6$, $5 \mapsto v_1$, $6 \mapsto v_2$, $7 \mapsto v_7$, $8 \mapsto v_{12}$, $9 \mapsto v_{11}$, $10 \mapsto v_{10}$, $11 \mapsto v_9$, $12 \mapsto v_8$, $a_1 \mapsto u_1$, $a_2 \mapsto u_2$. This completes computation of the DSEM for ≤ 15. This gives the following result.

4.3.2 Result

Theorem 4.3 *There exists a unique DSEM of type $[3^6 : 3^2,4,3,4]$ on the surfaces of Euler characteristic 0 for ≤ 15 vertices. This is $K_{1(2,12)}[3^6 : 3^2,4,3,4]$ on Klein bottle, given in Fig. 4.3.*

4.4 Computation and classification for type $[3^3,4^2 : 3^2,4,3,4]_1$

Consider the following DSEM of type $[3^3,4^2 : 3^2,4,3,4]_1$, shown in Fig. 4.4, on torus denoted by $T_{1(4,8)}[3^3,4^2 : 3^2,4,3,4]_1$.



$T_{1(4,8)}[3^3,4^2 : 3^2,4,3,4]_1$

Fig. 4.4: Doubley semi-equivelar map on torus of type $[3^3,4^2 : 3^2,4,3,4]_1$

4.4.1 Computation

Let M be a map of the type $[3^3,4^2 : 3^2,4,3,4]_1$ with the vertex set V. Let $V_{(3^3)}$ and $V_{(3^2,4,3,4)}$ denote the sets of vertices with face-sequence types $(3^3,4^2)$ and $(3^2,4,3,4)$, respectively. It is easy to see that $2|V_{(3^3,4^2)}| = |V_{(3^2,4,3,4)}|$ and $|V_{(3^2,4,3,4)}|$ is multiple of 4. Therefore, for $|V| \le 15$, we let $V = \{a_1, a_2, \ldots, a_{|V_{(3^3,4^2)}|}, 1, 2, \ldots, 2|V_{(3^3,4^2)}|\}$, where $|V_{(3^3,4^2)}| \le 4$. Assume that $\text{lk}(a_1) = C_7(2,3,[4,5,\boldsymbol{a}_2,6,1])$. Then $\text{lk}(a_2) = C_7(7,8,[5,4,\boldsymbol{a}_1,1,6])$. This implies $\text{lk}(2) = C_7(a_1,[1,x_1,x_2],[n_1,n_2,3])$ or $\text{lk}(2) = C_7(a_1,[3,x_1,x_2],[n_1,n_2,1])$, for $x_1,x_2 \in V_{(3^3,4^2)}$ and $n_1,n_2 \in V_{(3^2,4,3,4)}$. In the first case of lk(2), considering lk(1), we see three quadrangular faces incident at 1, which is not allowed. On the other hand when $\text{lk}(2) = C_7(a_1,[3,x_1,x_2],[n_1,n_2,1])$, we get $x_1 = a_3$, $x_2 = a_4$ and $(n_1,n_2) \in \{(5,8),(7,8),(8,5),(8,7)\}$.

For $(n_1, n_2) = (7, 8)$, considering lk(1) and lk(2), we see that $\text{lk}(a_3) = C_7(1, 8, [3, 2, \boldsymbol{a}_4, n_3, 6])$ or $\text{lk}(a_3) = C_7(1, 6, [3, 2, \boldsymbol{a}_4, n_3, 8])$, but for both the cases of $\text{lk}(a_3)$, we get no suitable value for n_3 in $V_{(3^2,4,3,4)}$. So $(n_1, n_2) \neq (7, 8)$.

For $(n_1, n_2) = (8, 5)$, then completing lk(2), lk(1) and proceeding, as in previous case, we see that lk(3) can not be completed.

For $(n_1, n_2) = (8, 7)$, considering lk(2) and lk(1), we see that lk(7) can not be completed.

For $(n_1, n_2) = (5, 8)$, successively, we get $\text{lk}(2) = C_7(a_1, [3, a_3, a_4], [5, 8, 1])$, $\text{lk}(5) = C_7(a_4, [4, a_1, a_2], [8, 1, 2])$, $\text{lk}(1) = C_7(a_3, [6, a_2, a_1], [2, 5, 8])$. Then $\text{lk}(a_3) = C_7(1, 8, [3, 2, \boldsymbol{a}_4, 7, 6])$ or $\text{lk}(a_3) = C_7(1, 6, [3, 2, \boldsymbol{a}_4, 7, 8])$.

When $\text{lk}(a_3) = C_7(1, 8, [3, 2, \boldsymbol{a}_4, 7, 6])$, completing $\text{lk}(a_4)$, we see that lk(7) can not be completed.

When $\text{lk}(a_3) = C_7(1, 6, [3, 2, \boldsymbol{a}_4, 7, 8])$, completing successively, we get $M \cong T_{1(4,8)}[3^3, 4^2 : 3^2, 4, 3, 4]_1$ by the map $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \leq i \leq 8$, $1 \leq j \leq 4$. Thus the computation is completed for ≤ 15 vertices. This leads to the following result.

4.4.2 Result

Theorem 4.4 *There exists a unique DSEM of type* $[3^3, 4^2 : 3^2, 4, 3, 4]_1$ *with number of vertices* ≤ 15. *This is* $T_{1(4,8)}[3^3, 4^2 : 3^2, 4, 3, 4]_1$ *on torus, shown in Fig. 4.4.*

4.5 Computation and classification for type $[3^3, 4^2 : 3^2, 4, 3, 4]_2$

Consider the following DSEM of type $[3^3, 4^2 : 3^2, 4, 3, 4]_2$, shown in Fig. 4.5, on Klein bottle denoted by $K_{1(6,6)}[3^3, 4^2 : 3^2, 4, 3, 4]_2$.

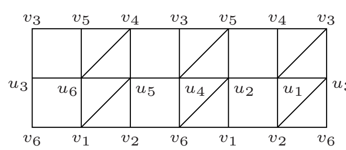


Fig. 4.5: Doubley semi-equivelar map of type $[3^3, 4^2 : 3^2, 4, 3, 4]_1$ on Klein bottle

4.5.1 Computation

Let M is a map of the type $[3^3, 4^2 : 3^2, 4, 3, 4]_2$ with the vertex set V. Let $V_{(3^3,4^2)}$ and $V_{(3^2,4,3,4)}$ denote the sets of vertices with face-sequence types $(3^3, 4^2)$ and $(3^2, 4, 3, 4)$ respectively. Then we see easily that $|V_{(3^3,4^2)}| = |V_{(3^2,4,3,4)}| = 2k$ for $k \in \mathbb{N}$. Thus for $|V| (= |V_{(3^3,4^2)}| + |V_{(3^2,4,3,4)}|) \leq 15$, we let $V = \{a_1, a_2, \ldots, a_{|V_{(3^3,4^2)}|}, 1, 2, \ldots, |V_{(3^3,4^2)}|\}$, where $|V_{(3^3,4^2)}| = 2k$ for $k \leq 3$. Assuming, without loss of generality, $\text{lk}(a_1) = C_7(a_3, 3, [4, 5, \boldsymbol{a}_2, 1, 2])$. This implies $\text{lk}(a_2) = C_7(x_1, n_1, [5, 4, \boldsymbol{a}_1, 2, 1])$ or $\text{lk}(a_2) = C_7(x_1, n_1, [1, 2, \boldsymbol{a}_1, 4, 5])$ for $x_1 \in V_{(3^3,4^2)}$ and $n_1 \in V_{(3^2,4,3,4)}$.

If $\text{lk}(a_2) = C_7(x_1, n_1, [5, 4, \boldsymbol{a}_1, 2, 1])$, then a small calculation shows that no such map exists for the given V. On the other hand, if $\text{lk}(a_2) = C_7(x_1, n_1, [1, 2, \boldsymbol{a}_1, 4, 5])$, then we have $x_1 = a_4$, $n_1 = 6$. This implies $\text{lk}(2) = C_7(a_3, [a_1, a_2, 1], [x_2, x_3, n_2])$, where $(x_2, x_3, n_2) \in \{(a_5, a_4, 5), (a_5, a_4, 6)\}$. If $(x_2, x_3, n_2) = (a_5, a_4, 5)$, then $\text{lk}(a_3) = C_7(a_1, 2, [5, n_3, \boldsymbol{a}_6, n_4, 3])$. Now considering lk(5) we see three quadrangular faces incident at 5, which is not allowed. If $(x_2, x_3, n_2) = (a_5, a_4, 6)$, $\text{lk}(2) = C_7(a_3, [a_1, a_2, 1], [a_5, a_4, 6])$. This implies $\text{lk}(a_3) = C_7(a_1, 2, [6, n_3, \mathbf{a_6}, n_4, 3])$, where $(n_3, n_4) \in \{(1, 5), (4, 1), (4, 5)\}$. In case $(n_3, n_4) = (4, 1)$ and $(4, 5)$, we see respectively lk(1) and lk(4) can not be completed. If $(n_3, n_4) = (1, 5)$, $M \cong K_{1(6,6)}[3^3, 4^2 : 3^2, 4, 3, 4]_2$ by the map $i \mapsto v_i$, $a_i \mapsto u_i$, $1 \leq i \leq 6$. Thus the computation is completed. Then we obtain the following result.

4.5.2 Result

Theorem 4.5 *There exists a unique DSEM of type* $[3^3, 4^2 : 3^2, 4, 3, 4]_2$ *with number of vertices* ≤ 15. *This is* $K_{1(6,6)}[3^3, 4^2 : 3^2, 4, 3, 4]_2$ *on Klein bottle, shown in Fig. 4.5.*

4.6 Computation and classification for type $[3^3, 4^2 : 4^4]_1$

Consider the DSEMs of type $[3^3, 4^2 : 4^4]_1$, in Fig. 4.6 on torus and Klein bottle denoted by $T_{i(n,2n)}[3^3, 4^2 : 4^4]_1$, for $i \in \{1, \ldots, 6\}$, and $K_{i(n,2n)}[3^3, 4^2 : 4^4]_1$, for $i \in \{1, \ldots, 3\}$, respectively.

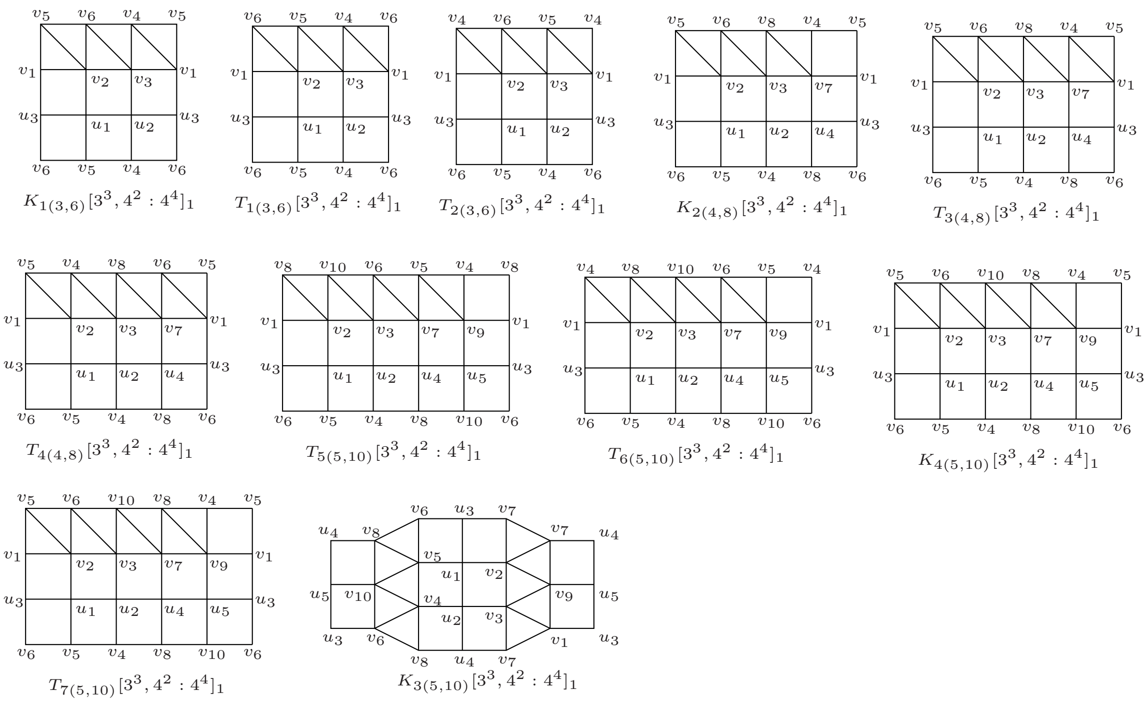


Fig. 4.6: Doubley semi-equivelar maps of type $[3^3, 4^2 : 4^4]_1$ on torus and Klein bottle

Claim 4.4 *For the maps above we have the following:*

(*g*) $T_{1(3,6)}[3^3, 4^2 : 4^4]_1 \ncong T_{2(3,6)}[3^3, 4^2 : 4^4]_1$.

(*h*) $T_{3(4,8)}[3^3, 4^2 : 4^4]_1 \ncong T_{4(4,8)}[3^3, 4^2 : 4^4]_1$.

(*i*) $T_{5(5,10)}[3^3, 4^2 : 4^4]_1 \ncong T_{6(5,10)}[3^3, 4^2 : 4^4]_1 \ncong T_{7(5,10)}[3^3, 4^2 : 4^4]_1$.

(*j*) $K_{3(5,10)}[3^3, 4^2 : 4^4]_1 \ncong K_{4(5,10)}$-$[3^3, 4^2 : 4^4]_1$.

Proof. Follows by considering the following polynomials:

$p(EG(T_{1(3,6)}[3^3, 4^2 : 4^4]_1)) = x^9 - 21x^7 - 24x^6 + 72x^5 + 72x^4 - 99x^3 - 54x^2 + 54x$,

$p(EG(T_{2(3,6)}[3^3, 4^2 : 4^4]_1)) = x^9 - 21x^7 - 18x^6 + 54x^5$,

$p(EG(T_{3(4,8)}[3^3, 4^2 : 4^4]_1)) = x^{12} - 28x^{10} - 16x_9 + 212x^8 + 88x^7 - 684x^6 - 48x^5 + 912x^4 - 272x^3 - 240x^2 + 96x$,

$p(EG(T_{4(4,8)}[3^3, 4^2 : 4^4]_1)) = x^{12} - 28x^{10} - 24x_9 + 212x^8 + 280x^7 - 524x^6 - 976x^5 + 80x^4 + 880x^3 + 528x^2 + 96x$,

$p(EG(K_{3(5,10)}[3^3, 4^2 : 4^4]_1)) = x^{15} - 5x^{13} - 40x^{12} + 385x^{11} + 790x^{10} - 1100x_9 - 3620x^8 + 55x^7 + 6200x^6 + 3305x^5 - 3500x^4 - 3265x^3 - 190x^2 + 270x - 24$,

$p(EG(K_{4(5,10)}[3^3, 4^2 : 4^4]_1)) = x^{15} - 35x^{13} - 24x^{12} + 401x^{11} + 434x^{10} - 1832x_9 - 2468x^8 + 3123x^7 + 5232x^6 - 939x^5 - 3716x^4 - 1261x^3 + 30x^2 + 30x$,

$p(EG(T_{5(5,10)}[3^3,4^2:4^4]_1)) = x^{15}-35x^{13}-20x^{12}+425x^{11}+294x^{10}-2500x_9-1520x^8+7855x^7+3060x^6-12919x^5-1100x^4+8815x^3-2250x^2+150x$,

$p(EG(T_{6(5,10)}[3^3,4^2:4^4]_1)) = x^{15}-35x^{13}-20x^{12}+395x^{11}+344x^{10}-1790x_9-1960x^8+3150x^7+3920x^6-2059x^5-3000x^4+235x^3+750x^2+150x$,

$p(EG(T_{7(5,10)}[3^3,4^2:4^4]_1)) = x^{15}-35x^{13}-30x^{12}+395x^{11}+594x^{10}-1495x_9-3360x^8+175x^7+3990x^6+2166x^5$.

4.6.1 Computation

Let M be a map of the type $[3^3,4^2:4^4]_1$ with the vertex set V. Let $V_{(4^4)}$ and $V_{(3^3,4^2)}$ denote the sets of vertices with face-sequence types (4^4) and $(3^3,4^2)$, respectively. Then counting the number of quadrangular faces in terms of $|V_{(3^3,4^2)}|$ and $|V_{(4^4)}|$ we see easily that $|V_{(3^3,4^2)}| = 2|V_{(4^4)}|$. Thus for $|V| \le 15$, we let $V = \{a_1, a_2, \dots, a_{|V_{(4^4)}|}, 1, 2, \dots, 2|V_{(4^4)}|\}$ such that $|V_{(4^4)}| \le 5$. Assume, without loss of generality, $\mathrm{lk}(a_1) = C_8(\boldsymbol{a}_3, 1, \boldsymbol{2}, 3, \boldsymbol{a}_2, 4, \boldsymbol{5}, 6)$. This implies $\mathrm{lk}(a_2) = C_8(\boldsymbol{a}_1, 2, \boldsymbol{3}, n_1, \boldsymbol{x}_1, n_2, \boldsymbol{4}, 5)$ for $x_1 \in V_{(4^4)}$ and $n_1, n_2 \in V_{(3^3,4^2)}$. Observe that $x_1 \in \{a_3, a_4\}$.
Case 1. When $x_1 = a_3$, then $(n_1,n_2) \in \{(1,6),(6,1)\}$.

If $(n_1,n_2) = (6,1)$, then $\mathrm{lk}(a_2) = C_8(\boldsymbol{a}_1, 2, \boldsymbol{3}, 6, \boldsymbol{a}_3, 1, \boldsymbol{4}, 5)$ and $\mathrm{lk}(a_3) = C_8(\boldsymbol{a}_2, 3, \boldsymbol{6}, 5, \boldsymbol{a}_1, 2, \boldsymbol{1}, 4)$. This implies $\mathrm{lk}(1) = C_7(n_3, n_4, [2, a_1, \boldsymbol{a}_3, a_2, 4])$. It is easy to see that $(n_3,n_4) \in \{(3,6),(5,6),(6,3),(6,5)\}$. But for these values of (n_3,n_4), we see easily that no map exists.

On the other hand, if $(n_1,n_2) = (1,6)$, then $\mathrm{lk}(a_2) = C_8(\boldsymbol{a}_1, 2, \boldsymbol{3}, 1, \boldsymbol{a}_3, 6, \boldsymbol{4}, 5)$, $\mathrm{lk}(a_3) = C_8(\boldsymbol{a}_2, 3, \boldsymbol{1}, 2, \boldsymbol{a}_1, 5, \boldsymbol{6}, 4)$. This implies $\mathrm{lk}(1) = C_7(n_3, n_4, [2, a_1, \boldsymbol{a}_3, a_2, 3])$, for $(n_3,n_4) \in \{(4,5),(4,6),(5,4),(5,6),(6,4),(6,5)\}$.

Observe that, $(5,6) \cong (4,5)$ by the map $(1,3,2)(4,5,6)(a_1,a_3,a_2)$; $(6,4) \cong (4,5)$ by the map $(1,2,3)(4,6,5)(a_1,a_2,\ a_3)$ and $(6,5) \cong (4,6)$ by the map $(2,3)(4,5)(a_1,a_2)$. Thus we search for $(n_3,n_4) \in \{(4,5),(4,6),(5,4),(5,6)\}$. Now doing computation for these cases, we see:

If $(n_3,n_4) = (4,5)$, $M \cong K_{1(3,6)}[3^3,4^2:4^4]_1$ by the map $i \mapsto v_6$, $a_j \mapsto u_j$, $1 \le i \le 6$, $1 \le j \le 3$.

If $(n_3,n_4) = (4,6)$, $M \cong T_{1(3,6)}[3^3,4^2:4^4]_1$ by the map $i \mapsto v_6$, $a_j \mapsto u_j$, $1 \le i \le 6$, $1 \le j \le 3$.

If $(n_3,n_4) = (5,4)$, $M \cong T_{2(3,6)}[3^3,4^2:4^4]_1$ by the map $i \mapsto v_6$, $a_j \mapsto u_j$, $1 \le i \le 6$, $1 \le j \le 3$.
Case 2. When $x_1 = a_4$, hen considering $\mathrm{lk}(a_3) = C_8(\boldsymbol{a}_1, 2, \boldsymbol{1}, n_3, \boldsymbol{x}_2, n_4, \boldsymbol{6}, 5)$ we get $x_2 \in \{a_4, a_5\}$.
Subcase 2.1. If $x_2 = a_4$, then $(n_3,n_4) \in \{(7,8),(8,7)\}$. If $(n_3,n_4) = (8,7)$, $(n_5,n_6) \in \{(4,5),(5,4),(5,6),(6,5),(6,7),(7,\ 3),(7,6)\}$. If $(n_5,n_6) = (4,5)$, then considering successively $\mathrm{lk}(1)$, $\mathrm{lk}(5)$ and $\mathrm{lk}(8)$ we see that $\deg(4) > 5$, a contradiction. If $(n_5,n_6) = (5,4)$, then considering successively, $\mathrm{lk}(1)$, $\mathrm{lk}(5)$, $\mathrm{lk}(2)$ and $\mathrm{lk}(6)$, we get $\mathrm{lk}(7)$ of length 5, a contradiction. Proceeding similarly for the rest of the cases of (n_5,n_6), we see easily that no map exists.

On the other hand, if $(n_3,n_4) = (7,8)$, then $\mathrm{lk}(1) = C_7(n_5, n_6, [2, a_1, \boldsymbol{a}_3, a_4, 7])$, where $(n_5,n_6) \in \{(4,5),(4,8),(5,4),(5,\ 6),(6,5),(6,8),(8,4),(8,6)\}$. But, $(5,4) \cong (4,8)$ by the map $(1,3)(4,6)(a_2,a_3)$; $(5,6) \cong (4,5)$ by the map $(1,7)(2,3)(4,5)\ (6,8)(a_1,a_2)(a_3,a_4)$; $(6,8) \cong (4,5)$ by the map $(1,2,3,7)(4,8,6,5)(a_1,a_2,a_4,a_3)$; $(8,4) \cong (4,5)$ by the map $(1,3)(2,7)(4,\ 6)(5,8)(a_1,a_4)\ (a_2,a_3)$; $(8,6) \cong (6,5)$ by the map $(1,7)(2,3)(4,5)(6,8)(a_1,a_2)(a_3,a_4)$. So we search for $(n_5,n_6) \in \{(4,5),(4,8),(6,5)\}$. Now doing computation for these cases, we see:

If $(n_5,n_6) = (4,5)$, $M \cong K_{2(4,8)}[3^3,4^2:4^4]_1$ by the map $i \mapsto v_6$, $a_j \mapsto u_j$, $1 \le i \le 8$, $1 \le j \le 4$.

If $(n_5,n_6) = (4,8)$, $M \cong T_{3(4,8)}[3^3,4^2:4^4]_1$ by the map $i \mapsto v_6$, $a_j \mapsto u_j$, $1 \le i \le 8$, $1 \le j \le 4$.

If $(n_5,n_6) = (6,5)$, $M \cong T_{4(4,8)}[3^3,4^2:4^4]_1$ by the map $i \mapsto v_6$, $a_j \mapsto u_j$, $1 \le i \le 8$, $1 \le j \le 4$.
Subcase 2.2. When $x_2 = a_5$, then considering $\mathrm{lk}(a_4)$, we see that $(n_5,n_6) \in \{(9,10),(10,9)\}$. If $(n_5,n_6) = (10,9)$, $(n_7,n_8) \in \{(3,7),(4,5),(4,8),(5,4),(5,6),(6,5),(6,10),(7,10),(10,6),(10,\ 7)\}$. But, a small calculation shows that no map exists for these values of (n_7,n_8).

While for $(n_5,n_6) = (9,10)$ completing successively $\mathrm{lk}(a_4)$, $\mathrm{lk}(a_5)$ we get $\mathrm{lk}(1) = C_7(n_7, n_8, [2, a_1, \boldsymbol{a}_3, a_5, 9])$. Then, up to isomorphism, we get $(n_7,n_8) \in \{(3,7),(4,5),\ (5,4),(6,5)\}$. Doing computation for these cases, we see:

If $(n_7,n_8) = (3,7)$, $M \cong K_{3(5,10)}[3^3,4^2:4^4]_1$ by the map $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 10$, $1 \le j \le 5$.

If $(n_7,n_8) = (4,5)$, $M \cong K_{4(5,10)}[3^3,4^2:4^4]_1$ by the map $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 10$, $1 \le j \le 5$.

If $(n_7, n_8) = (4, 8)$, $M \cong T_{5(5,10)}[3^3, 4^2 : 4^4]_1$ by the map $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 10$, $1 \le j \le 5$.
If $(n_7, n_8) = (5, 4)$, $M \cong T_{6(5,10)}[3^3, 4^2 : 4^4]_1$ by the map $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 10$, $1 \le j \le 5$.
If $(n_7, n_8) = (6, 5)$, $M \cong T_{7(5,10)}[3^3, 4^2 : 4^4]_1$ by the map $i \mapsto v_i$, $a_j \mapsto u_j$, $1 \le i \le 10$, $1 \le j \le 5$.
This completes the computation and we get the following results.

4.6.2 Results

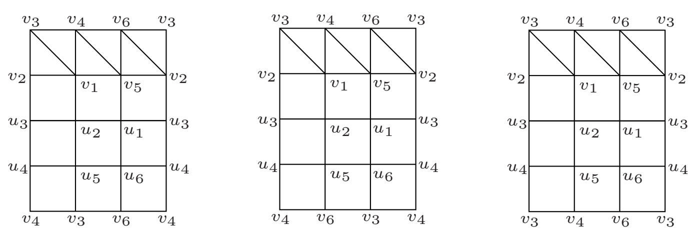
Lemma 4.3 *Let M be a DSEM of type $[3^3, 4^2 : 4^4]_1$ with number of vertices ≤ 15. Then M is isomorphic to one of the following: $T_{1(3,6)}[3^3, 4^2 : 4^4]_1$, $T_{2(3,6)}[3^3, 4^2 : 4^4]_1$, $K_{1(3,6)}[3^3, 4^2 : 4^4]_1$, $T_{3(4,8)}[3^3, 4^2 : 4^4]_1$, $T_{4(4,8)}[3^3, 4^2 : 4^4]_1$, $K_{2(4,8)}[3^3, 4^2 : 4^4]_1$, $T_{5(5,10)}[3^3, 4^2 : 4^4]_1$, $T_{6(5,10)}[3^3, 4^2 : 4^4]_1$, $T_{7(5,10)}[3^3, 4^2 : 4^4]_1$, $K_{3(5,10)}[3^3, 4^2 : 4^4]_1$, $K_{4(5,10)}[3^3, 4^2 : 4^4]_1$, shown in Fig. 4.6.*

Combining the above lemma together with the Claim 4.4, it follows that:

Theorem 4.6 *There are exactly 11 non-isomorphic DSEMs of type $[3^3, 4^2 : 4^4]_1$ on the surfaces of Euler characteristic 0 with ≤ 15 vertices. Out of these 7 are on torus and remaining 4 are on Klein bottle.*

4.7 Computation and classification for type $[3^3, 4^2 : 4^4]_2$

Consider the following DSEMs of type $[3^3, 4^2 : 4^4]_2$, shown in Fig. 4.7, on torus and Klein bottle denoted by $T_{i(6,6)}[3^3, 4^2 : 4^4]_2$, for $i = 1, 2$, and $K_{1(6,6)}[3^3, 4^2 : 4^4]_2$, respectively.



$K_{1(6,6)}[3^3, 4^2 : 4^4]_2$ $T_{1(6,6)}[3^3, 4^2 : 4^4]_2$ $T_{2(6,6)}[3^3, 4^2 : 4^4]_2$

Fig. 4.7: Doubley semi-equivelar maps of type $[3^3, 4^2 : 4^4]_2$ on torus and Klein bottle

Claim 4.5 $T_{1(6,6)}[3^3, 4^2 : 4^4]_2 \ncong T_{2(6,6)}[3^3, 4^2 : 4^4]_2$.

Proof. See the following polynomials:
$p(EG(T_{1(6,6)}[3^3, 4^2 : 4^4]_2)) = x^{12} - 27x^{10} - 20x_9 + 207x^8 + 168x^7 - 610x^6 - 288x^5 + 723x^4 - 136x^3 - 171x^2 + 84x - 11$,

$p(EG(T_{2(6,6)}[3^3, 4^2 : 4^4]_2)) = x^{12} - 27x^{10} - 20x_9 + 201x^8 + 192x^7 - 532x^6 - 552x^5 + 492x^4 + 560x^3 - 84x^2 - 192x - 44$.

4.7.1 Computation

Let M be a map of the type $[3^3, 4^2 : 4^4]_2$ with the vertex set V. Let $V_{(4^4)}$ and $V_{(3^3,4^2)}$ denote the sets of vertices with face-sequence types (4^4) and $(3^3, 4^2)$, respectively. Then, it is easy to see that M has the number of quadrangular faces $\frac{3|V_{(4^4)}|}{2}$ or $(|V_{(3^3,4^2)}| + \frac{|V_{4^4}|}{2})$. This implies $|V_{(4^4)}| = |V_{(3^3,4^2)}| = 2k$ for $k \in \mathbb{N}$. Therefore for $|V| \le 15$, we let $V = \{a_1, a_2, \ldots, a_{|V_{(4^4)}|}, 1, 2, \ldots, |V_{(4^4)}|\}$, where $|V_{(4^4)}| = 2k$ for $k \le 3$. Without loss of generality, we may assume $\mathrm{lk}(1) = C_7(3, 4, [5, a_1, \boldsymbol{a}_2, a_3, 2])$. Then $\mathrm{lk}(a_2) = C_8(\boldsymbol{a}_3, 2, \mathbf{1}, 5, \boldsymbol{a}_1, a_6, \boldsymbol{a}_5, a_4)$, $\mathrm{lk}(a_1) = C_8(\boldsymbol{a}_2, 1, \mathbf{5}, n_1, \boldsymbol{x}_1, x_2, \boldsymbol{a}_6, a_5)$ for $n_1 \in V_m$ and $x_1, x_2 \in V_l$. Observe that $(n_1, x_1, x_2) \in \{(2, a_3, a_4), (3, a_4, a_3), (6, a_4, a_3)\}$.
Case 1. When $(n_1, x_1, x_2) = (3, a_4, a_3)$ then considering successively $\mathrm{lk}(a_1)$, $\mathrm{lk}(a_4)$, $\mathrm{lk}(a_5)$, $\mathrm{lk}(3)$ and $\mathrm{lk}(4)$ we see that $\mathrm{lk}(5)$ can not be completed. So $(n_1, x_1, x_2) \ne (2, a_4, a_3)$.
Case 2. When $(n_1, x_1, x_2) = (6, a_4, a_3)$ then $\mathrm{lk}(a_1) = C_8(\boldsymbol{a}_2, 1, \mathbf{5}, 6, \boldsymbol{a}_4, a_3, \boldsymbol{a}_6, a_5)$ and $\mathrm{lk}(a_4) = C_8(\boldsymbol{a}_1, 5, \mathbf{6}, n_2, \boldsymbol{a}_5, a_2, \boldsymbol{a}_3, a_6)$ for $n_2 \in \{3, 4, 7\}$.

If $n_2 = 3$ then considering successively lk(a_4), lk(a_5), lk(a_6), lk(a_3), lk(3) and lk(2) we see lk(4) can not be completed.

If $n_2 = 4$ then considering successively lk(a_4), lk(a_5), lk(a_6), lk(a_3), lk(4), as in previous case, we see that lk(5) can not be completed.

If $n_2 = 7$ then considering successively lk(a_4), lk(a_5) and lk(a_6) we see that lk(a_3) can not be completed. So $(n_1, x_1, x_2) \neq (6, a_4, a_3)$.

Case 3. If $(n_1, x_1, x_2) = (2, a_3, a_4)$, then successively completing lk(a_1), lk(a_3), lk(2) and lk(5) we get lk(4) $= C_7(1, 5, [6, x_5, \boldsymbol{x}_4, x_3, 3])$ for $(x_3, x_4, x_5) \in \{(a_5, a_4, a_6), (a_6, a_4, a_5), (a_4, a_5, a_6), (a_6, a_5, a_4), (a_4, a_6, a_5), (a_5, a_6, a_4)\}$.

Note that $(a_6, a_5, a_4) \cong (a_5, a_4, a_6)$ by the map $(1, 5, 2)(3, 4, 6)(a_1, a_3, a_2)(a_4, a_5, a_6)$; $(a_4, a_6, a_5) \cong (a_5, a_4, a_6)$ by the map $(1, 2, 5)$ $(3, 6, 4)(a_1, a_2, a_3)(a_4, a_6, a_5)$; $(a_5, a_6, a_4) \cong (a_4, a_5, a_6)$ by the map $(2, 5)(3, 4)(a_1, a_3)(a_4, a_6)$.

Thus we do computation for $(x_3, x_4, x_5) \in \{(a_5, a_4, a_6), (a_6, a_4, a_5), (a_4, a_5, a_6)\}$. This gives:

If $(x_3, x_4, x_5) = (a_5, a_4, a_6)$, $M \cong K_{1(6,6)}[3^3, 4^2 : 4^4]_2$ by the map $i \mapsto v_i$, $a_i \mapsto u_i$, $1 \leq i \leq 6$.

If $(x_3, x_4, x_5) = (a_6, a_4, a_5)$, $M \cong T_{1(6,6)}[3^3, 4^2 : 4^4]_2$ by the map $i \mapsto v_i$, $a_i \mapsto u_i$, $1 \leq i \leq 6$.

If $(x_3, x_4, x_5) = (a_4, a_5, a_6)$, $M \cong T_{2(6,6)}[3^3, 4^2 : 4^4]_2$ by the map $i \mapsto v_i$, $a_i \mapsto u_i$, $1 \leq i \leq 6$.

Thus the computation is completed and we get the following results.

4.7.2 Results

Lemma 4.4 *Let M be a DSEM of type $[3^3, 4^2 : 4^4]_2$ with number of vertices ≤ 15. Then M is isomorphic to $T_{1(6,6)}[3^3, 4^2 : 4^4]_2$, $T_{2(6,6)}[3^3, 4^2 : 4^4]_2$ or $K_{1(6,6)}[3^3, 4^2 : 4^4]_2$, shown in Fig. 4.7.*

Combining the Lemma 4.4 together with the Claim 4.5, it follows that:

Theorem 4.7 *There are exactly 3 non-isomorphic DSEMs of type $[3^3, 4^2 : 4^4]_2$ on the surfaces of Euler characteristic 0 with ≤ 15 vertices. Out of these two are on torus and remaining on Klein bottle.*

5 Summary

From the theorems 4.1 - 4.7, it follows that:

Theorem 5.1 *There are at least* 35 *non-isomorphic DSEMs on the surfaces of Euler characteristic* 0 *with* ≤ 15 *vertices. Out of these,* 20 *are on the torus and remaining* 15 *are on the Klein bottle.*

A tabular form of the results obtained here is presented in the next page.

Table 5: DSEMs of face-size 4 on torus and Klein bottle on ≤ 15 vertices

S.No.	Map Type	$\lvert V \rvert$	$No.of maps$	On Torus	On Klein bottle
1.	$[3^6 : 3^3, 4^2]_1$	9	4	$T_{1(3,6)}[3^6{:}3^3, 4^2]_1$, $T_{2(3,6)}[3^6{:}3^3, 4^2]_1$	$K_{1(3,6)}[3^6{:}3^3, 4^2]_1$, $K_{2(3,6)}[3^6{:}3^3, 4^2]_1$
		12	6	$T_{3(4,8)}[3^6{:}3^3, 4^2]_1$, $T_{4(4,8)}[3^6{:}3^3, 4^2]_1$, $T_{5(4,8)}[3^6{:}3^3, 4^2]_1$	$K_{3(4,8)}[3^6{:}3^3, 4^2]_1$, $K_{4(4,8)}[3^6{:}3^3, 4^2]_1$, $K_{5(4,8)}[3^6{:}3^3, 4^2]_1$
		15	5	$T_{6(5,10)}[3^6{:}3^3, 4^2]_1$, $T_{7(5,10)}[3^6{:}3^3, 4^2]_1$, $T_{8(5,10)}[3^6{:}3^3, 4^2]_1$	$K_{6(5,10)}[3^6{:}3^3, 4^2]_1$, $K_{7(5,10)}[3^6{:}3^3, 4^2]_1$
2.	$[3^6 : 3^3, 4^2]_2$	12	3	$T_{1(6,6)}[3^6{:}3^3, 4^2]_2$, $T_{2(6,6)}[3^6{:}3^3, 4^2]_2$	$K_{1(6,6)}[3^6{:}3^3, 4^2]_2$

3.	$[3^6{:}3^2,4,3,4]$	14	1	-	$K_{1(2,12)}[3^6{:}3^2,4,3,4]$
4.	$[3^3.4^2{:}3^2,4,3,4]_1$	12	1	$T_{1(4,8)}[3^3,4^2{:}3^2,4,3,4]_1$	-
5.	$[3^3,4^2{:}3^2,4,3,4]_2$	12	1	-	$K_{1(6,6)}[3^3,4^2{:}3^2,4,3,4]_2$
6.	$[3^3,4^2:4^4]_1$	9	3	$T_{1(3,6)}[3^3,4^2{:}4^4]_1$, $T_{2(3,6)}[3^3,4^2{:}4^4]_1$	$K_{1(3,6)}[3^3,4^2{:}4^4]_1$
		12	3	$T_{3(4,8)}[3^3,4^2{:}4^4]_1$, $T_{4(4,8)}[3^3,4^2{:}4^4]_1$	$K_{2(4,8)}(3^3,4^2{:}4^4)_1$
		15	5	$T_{5(5,10)}[3^3,4^2{:}4^4]_1$, $T_{6(5,10)}[3^3,4^2{:}4^4]_1$, $T_{7(5,10)}[3^3,4^2{:}4^4]_1$	$K_{3(5,10)}[3^3,4^2{:}4^4]_1$, $K_{4(5,10)}[3^3,4^2{:}4^4]_1$
7.	$[3^3,4^2{:}4^4]_2$	12	3	$T_{1(6,6)}[3^3,4^2{:}4^4]_2$, $T_{2(6,6)}[3^3,4^2{:}4^4]_2$	$K_{1(6,6)}[3^3,4^2{:}4^4]_2$

6 Discussion

In [16], the authors constructed infinite series of semi-equivelar maps on torus and Klein bottle from equivelar maps by using elementary map operations: truncation and subdivision (these operations do not effect the symmetry of a map). Here, we present infinite series of the seven type doubly semi-equivelar maps for torus, one can explore similarly for Klein bottle.

Infinite series of DSEMs of types $[3^6 : 3^3,4^2]_1$, $[3^3,4^2 : 4^4]_1$, $[3^3,4^2 : 4^4]_2$, $[3^3,4^2 : 3^2,4,3,4]_2$, $[3^6 : 3^3,4^2]$ are constructed from infinite series of semi-equivelar map of type $[4^4]$ by subdividing the quadrangular faces as shown in the Fig. 6.1, Fig. 6.2, Fig. 6.3, Fig. 6.4 and Fig. 6.5 respectively. Infinite series of DSEM of type $[3^6 : 3^2,4,3,4]$ is obtained from an infinite series of semi-equivelar map of type $[6^3]$ by subdividing the hexagonal faces (by introducing a new vertex and joining it to the six vertices of the face by an edge) as shown in Fig. 6.6.

Although, we present infinite series of DSEM of type $[3^3,4^2 : 3^2,4,3,4]$, see Fig. 6.7. However we do not know whether this DSEM can be obtained from any semi-equivelar map by the above elementary map operations. This observation leads to the following question.

Question 1 *Can we obtain every doubly semi-equivelar map (corresponding to the 2-uniform tilings) on torus and Klein bottle from semi-equivelar maps (corresponding to the Archimedean tilings) by applying finite sequence of map operations on the same surface.*

Infinite series of DSEMs on torus:

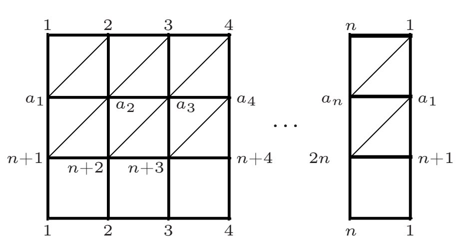


Fig. 6.1: DSEM of type-$[3^6{:}\ 3^3,4^2]_1 :\ (n \geq 3)$

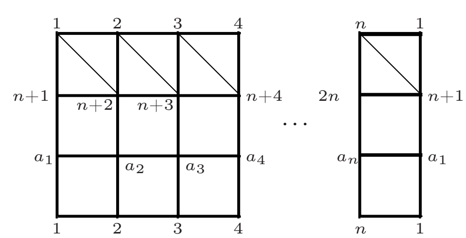


Fig. 6.2: DSEM of type-$[3^3,4^2{:}\ 4^4]_1 :\ (n \geq 3)$

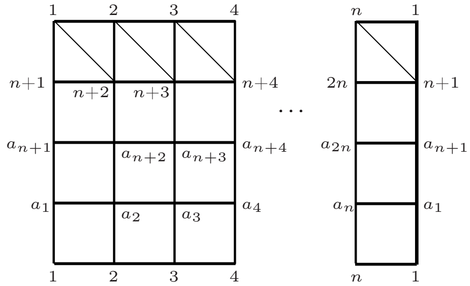


Fig. 6.3: DSEM of type-$[3^3, 4^2\colon\ 4^4]_2\colon\ (n \geq 3)$

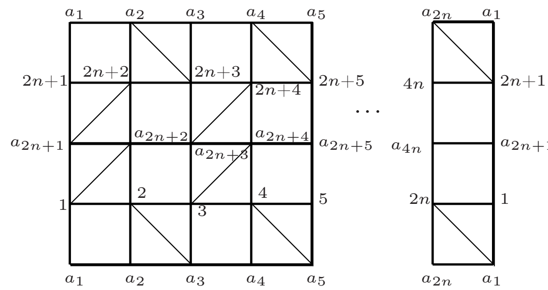


Fig. 6.4: DSEM of type-$[3^3, 4^2\colon\ 3^2, 4, 3, 4]_2\colon\ (n \geq 2)$

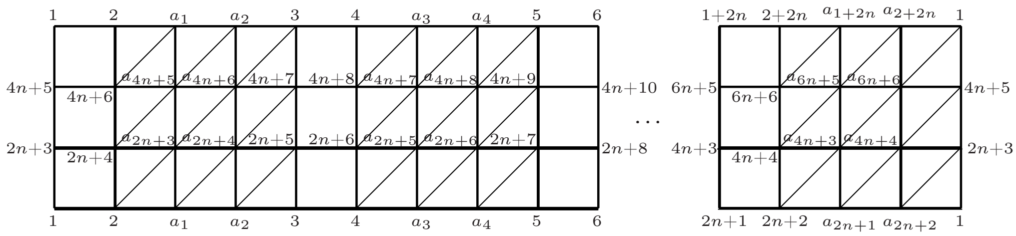


Fig. 6.5: DSEM of type-$[3^6 : 3^3, 4^2]_2\colon\ (n \geq 0)$

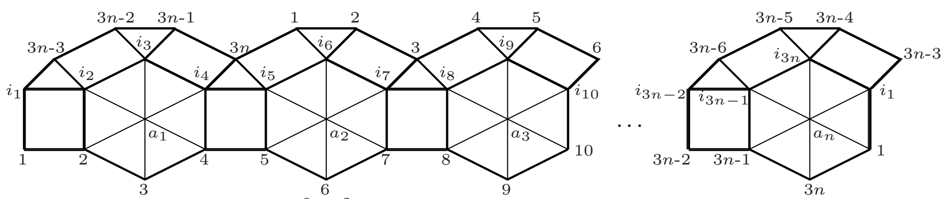


Fig. 6.6: DSEM of type -$[3^6 : 3^2, 4, 3, 4]\colon\ (n \geq 2)$

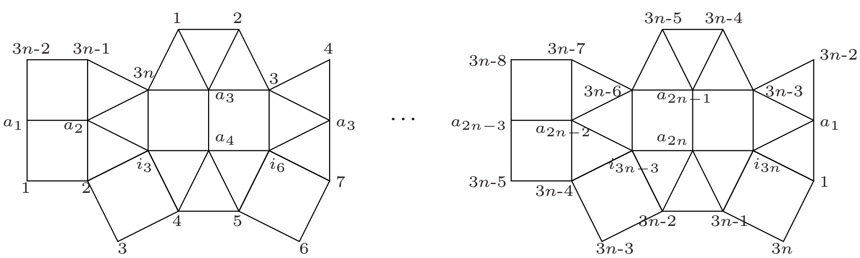


Fig. 6.7: DSEM of type-$[3^3, 4^2 : 3^2, 4, 3, 4]_1\colon\ (n \geq 3)$

If we study group structures associated to the maps, we see that DSEMs, obtained here, on torus are 2-uniform. For example, in case of type $[3^6 : 3^3, 4^2]_1$, we see that the groups:

$G_1 =< (v_1, v_3, v_5, v_4, v_2, v_6)(u_1, u_3, u_2) >$, $G_2 =< (v_1, v_2, v_5)(v_3, v_6, v_4)(u_1, u_3, u_2), (v_1, v_3)(v_2, v_4)(v_5, v_6)(u_2,\ u_3) >$, $G_3 =< (v_1, v_2, v_7, v_5)(v_3, v_8, v_6, v_4)(u_1, u_3, u_4, u_2), (v_1, v_4)(v_2, v_6)(v_3, v_5)(v_7, v_8)(u_1, u_2)(u_3,\ u_4) >$, $G_4 =< (v_1, v_5, v_7, v_2)\ (v_3, v_4, v_6, v_8)(u_1, u_2, u_4, u_3), (v_1, v_8)(v_2, v_6)(v_3, v_5)(v_4, v_7)(u_1, u_4)(u_2, u_3) >$, $G_5 =< (v_1, v_5, v_7, v_2)(v_3, v_4, v_6, v_8)(u_1, u_2, u_4, u_3), (v_1, v_4)(v_2, v_3)(v_5, v_6)(v_7, v_8) >$, $G_6 =< (v_1, v_2, v_7, v_{10}, v_5)(v_3, v_8, v_9, v_6, v_4)(u_1, u_3, u_5, u_4, u_2), (v_1, v_9)(v_2, v_8)(v_3, v_7)(v_4, v_{10})(v_5, v_6)(u_1, u_5)(u_2,\ u_4) >$, $G_7 =< (v_1, v_7, v_5, v_2, v_{10})(v_3, v_9, v_4, v_8, v_6)(u_1, u_5, u_2, u_3, u_4), (v_1, v_3)(v_2, v_4)(v_5, v_8)(v_6, v_7)(v_9, v_{10})(u_2, u_3)(u_4, u_5) >$ and $G_8 =< (v_1, v_{10}, v_2, v_5, v_7)(v_3, v_6, v_8, v_4, v_9)(u_1, u_4, u_3, u_2, u_5), (v_1, v_4)(v_2, v_3)(v_5, v_6)(v_7, v_8)(v_9, v_{10}) >$ act on the maps:

$T_{1(3,6)}[3^6 : 3^3, 4^2]_1$, $T_{2(3,6)}[3^6 : 3^3, 4^2]_1$, $T_{3(4,8)}[3^6 : 3^3, 4^2]_1$, $T_{4(4,8)}[3^6 : 3^3, 4^2]_1$, $T_{5(4,8)}[3^6 : 3^3, 4^2]_1$, $T_{6(5,10)}[3^6 : 3^3, 4^2]_1$, $T_{7(5,10)}[3^6 : 3^3, 4^2]_1$ and $T_{8(5,10)}[3^6 : 3^3, 4^2]_1$ respectively, such that under the action, the maps have two orbits of vertices. Similarly, we can easily find a group for the DSEMs of types: $[3^6 : 3^3, 4^2]_2$, $[3^3, 4^2 : 3^2, 4, 3, 4]_1$, $[3^3, 4^2 : 4^4]_1$ and $[3^3, 4^2 : 4^4]_2$ on torus, under which the maps have two orbits of vertices. However, this fact does not hold for the DSEMs on Klein bottle. For example, if we let $K_{1(3,6)}[3^6 : 3^3, 4^2]_1$, we get no automorphism which sends v_1 to v_2. This can be seen as follows: Suppose there is $f \in Aut(K_{1(3,6)}[3^6 : 3^3, 4^2]_1)$ such that $f(v_1) = v_2$. Then considering $\mathrm{lk}(v_1)$ and $\mathrm{lk}(v_2)$, we see, either $f(v_5) = v_5$ or $f(v_5) = v_1$. In the first case when $f(v_5) = v_5$, we get $f(u_2) = u_3$, $f(u_1) = u_1$, $f(v_2) = v_1$, $f(v_3) = v_6$, $f(v_6) = v_3$ and $f(v_4) = v_4$. Now if we see $\mathrm{lk}(u_1)$, we get a contradiction of the facts $f(u_1) = u_1$ and $f(v_3) = v_6$, as $v_6 \notin \mathrm{lk}(\mathrm{u}_1)$. So $f(v_5) \neq v_5$. Similarly, we see that $f(v_5) \neq v_1$. Combining these, we see that $f(v_1) \neq v_2$. This shows that $K_{1(3,6)}[3^6 : 3^3, 4^2]_1$ is not 2-uniform. This observation leads to ask the following question:

Question 2 *Are the doubly semi-equivelar maps (corresponding to the 2-uniform tilings) on torus 2-uniform?*

7 Conclusions

In this article, the notion of doubly semi-equivelar maps (DSEMs) has been introduced for the first time. A methodology has been presented to enumerate doubly semi-equivelar maps on torus and Klein bottle corresponding to the 2-uniform tilings $[3^6 : 3^3, 4^2]_1$, $[3^6 : 3^3, 4^2]_2$, $[3^6 : 3^2, 4, 3, 4]$, $[3^3, 4^2 : 3^2, 4, 3, 4]_1$, $[3^3, 4^2 : 3^2, 4, 3, 4]_2$, $[3^3, 4^2 : 4^4]_1$, $[3^3, 4^2 : 4^4]_2$. The methodology has been demonstrated to enumerate the DSEMs on at most 15 vertices. The enumeration provides at least 35 non-isomorphic DSEMs on the surfaces of Euler characteristic zero, out of these 20 are on torus and remaining 15 are on Klein bottle. Further, infinite series of these types DSEMs have been constructed. We know that a study of maps become more significant when certain symmetry involves, in view of this, the notion of 2-uniform maps (parallel to the notion of vertex-transitive maps for equivelar or semi-equivelar maps) has been introduced. During computation, it has been found that all the maps obtained on torus are 2-uniform, which does not hold in case of DSEMs on Klein bottle. This motivates us to explore the fact whether all the DSEMs on torus are 2-uniform. In literature, vertex-transitive maps have been studied extensively. It would be interesting to study 2-uniform maps not only for torus and Klein bottle but also for other close surfaces and to explore the analog notions of vertex-transitive maps for 2-uniform maps.

8 Data Availability

No data were used to support this study.

9 Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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